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# The impact of a mathematics research experience on teachers' conceptions of student learning

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THE IMPACT OF A MATHEMATICS RESEARCH EXPERIENCE ON  
TEACHERS' CONCEPTIONS OF STUDENT LEARNING

BY

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DISSERTATION

Submitted to the University of New Hampshire  
In Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in

Mathematics Education

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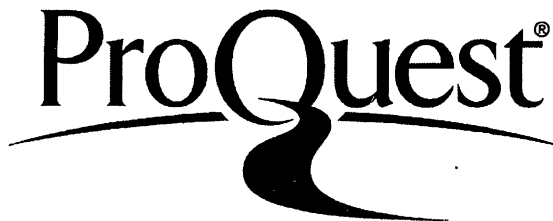
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## DEDICATION

This work is dedicated to the memory of my father,

*Kenneth Wayne Abel*

and to my mother,

*Ann Todd Abel*

My first teachers, they showed me through their example what it is to be a good teacher and a good person, how to ask meaningful questions, and to value the knowledge and experience of everyone I encounter. This work is worthwhile if it makes them proud.

It is also dedicated to my wife,

*Elizabeth Renfro Abel*

whose love and patience made it possible.

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## ABSTRACT

### THE IMPACT OF A MATHEMATICS RESEARCH EXPERIENCE ON TEACHERS' CONCEPTIONS OF STUDENT LEARNING

by

Todd Abel

University of New Hampshire, September, 2010

Many mathematics teacher professional development programs have either incorporated or been organized around a goal of providing “research-like” (Cuoco, 2001) experiences. That is, teachers participate in a project that somehow simulates the mathematics research process. Though some research studies have shown positive outcomes from such programs, researchers have cautioned against assuming universally positive benefits without sufficient evidence (Proulx and Bednarz, 2001). Teacher conceptions of student learning play an important role in lesson development and preparation for classroom work (Penso & Shoham, 2003). Similarities between the processes of mathematics research and student learning (Dreyfus, 1991) beg the question of whether experience with one (mathematics research) might impact the way one thinks about the other (student learning). The current study investigates the impact of one “research-like” professional development program on teachers’ conceptions of student learning.

This study used belief surveys combined with five case studies. The case studies were based on a series of task-based interviews utilizing lesson planning

tasks that employed Simon's (1995) notion of a *hypothetical learning trajectory*. The results indicate that teachers' primary beliefs remained consistent and impacted the ways in which they interpreted their experiences, but that some peripheral beliefs changed. General themes included an increased emphasis on exploring multiple problems in order to motivate conjectures or generalizations and increased empathy toward students learning unfamiliar content. Individual teachers exhibited some idiosyncratic changes, as well. For each individual, changes in peripheral beliefs were consistent with those aspects of the teacher's own learning experiences that he or she found to be most meaningful. Indeed, the results indicate that experience learning unfamiliar mathematics content was the aspect of the program that most powerfully impacted the participants. Teachers drew parallels between mathematics research and student learning, but only as they drew parallels between their own experience, which they understood to be "research-like", and that of their students. The implications of these results and the directions they suggest for future research are also explored.

## CHAPTER I

### INTRODUCTION AND BACKGROUND

#### **Introduction and Research Questions**

##### **Introduction**

Cuoco (2001) wrote, “there are very few absolutes in education, but one thing of which I am absolutely certain: *The best high school teachers are those who have a research-like experience in mathematics*” (p. 169, italics in original). His claim is based on many years of experience as a teacher, teacher educator, and curriculum developer, not on any rigorous research program. It would, however, be near-sighted to dismiss those years of experience. Cuoco makes a provocative, if unproven, claim, one that helped lead me to the research study described below. It prompts one to question whether or not his statement is true. The adjective “best” is difficult to define and therefore problematic, but it is worthwhile to consider the value of what Cuoco terms “research-like” experiences. At a basic level, that is what the study described herein investigated.

Programs that attempt to simulate mathematics research have gained increasing popularity recently in response to movements in mathematics education and national recommendations that emphasize the importance of teaching the process of mathematical thinking (CBMS, 2001; NCTM, 2000;

Skemp, 1987; Tall, 1991). Since the quintessential exemplar of mathematical thinking is the research mathematician, it is logical to consider the mathematics research process as a model for the process of mathematical thinking, and such programs do so. However, an emphasis on the behaviors and actions of mathematics professionals has pushed issues concerning the desirability and reasonableness of encouraging students to “act like mathematicians” (Watson, 2008, p. 3) into the spotlight. Though there is disagreement over whether having students “act like mathematicians” is actually possible (cf. Mendick, 2008; Proulx & Bednarz, 2009; Watson, 2008; Zazkis, 2008), there is significant agreement that teachers should encourage many of the ways of thinking (Harel & Sowder, 2005) and habits of mind (Cuoco, Goldenberg, & Mark, 1996) that are characteristic of mathematicians. Furthermore, there is some agreement that teachers should be equipped to foster environments where such flexible means of learning and doing are encouraged, and that research should focus on how they might be so equipped (Cuoco, 2001; National Research Council, 1999; NCTM, 1991; Proulx & Bednarz, 2009).

Several teacher education programs (both pre-service teacher education programs and in-service professional development) have taken as a point of emphasis that, in order to lead students in developing useful, flexible ways of thinking about mathematics in an exploratory setting, teachers must have experience developing such reasoning skills for themselves in a similar way. This has led to a professional development model where teachers are immersed in mathematical practices through problem explorations. In particular, several

researchers and teacher educators have created experiences for teachers that simulate mathematics research in order to put teachers in situations that necessitate reasoning similar to that of a mathematician<sup>1</sup> (cf., Badertscher, 2007; Chazan, et al., 2007; McCrone, Langrall, El-Zanati, & Mooney, 2008; Schifter & Fosnot, 1993; Smith, 2008; Stevens, 2001; Stevens, Cuoco, Burrill, Lewis, & McCallum, 2008). These are the “research-like” experiences Cuoco (2001) referred to in the quote above. Such programs, which I term *mathematics immersion experiences*, have been shown to impact the beliefs teachers hold about the nature of mathematics (McCrone, et al., 2008; Langrall, El-Zanati, & Mooney, 2008), teachers’ affect toward mathematics (Badertscher, 2007; Chazan, et al., 2006; Davis & Hersh, 1981; Stevens, 2001; Stevens, et al., 2008), and to lead to alterations in teaching style (Stevens, 2001; Stevens, et al., 2008). That evidence suggests that these mathematics immersion experiences have some impact on teachers. But, as Proulx and Bednarz (2009) have noted, “unless and until we know more, we have to be careful around the assumption that living ‘mathematically genuine’ experiences will change teachers’ practices” (p. 28).

If more must be known about the impact “mathematically genuine” experiences have on teachers, where precisely should one look? There are many possible answers to that question, but ultimately, investigations should try to determine if students are better equipped mathematically as a result of teacher

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<sup>1</sup> Indeed, some have called for this to be an integral part of teacher education programs (Cuoco, 2001; National Research Council, 1999).

participation in such an experience. This is difficult to ascertain without long-term and large-scale studies that were a practical impossibility in this case. It is reasonable, however, to consider aspects of teacher conceptions (including beliefs, values, and knowledge) that have some impact in determining practice. Teacher conceptions of student learning are one area satisfying that requirement.

A significant body of research has shown that teacher conceptions of student learning shape lesson planning and implementation (see Beswick, 2007; Escudero & Sánchez, 2007; Penso & Shoham, 2003; Simon, 1995) as well as curriculum implementation (Lloyd & Wilson, 1998). Others have shown that attending to student understandings influences teacher beliefs and practice (cf., Carpenter, Fennema, & Franke, 1996). Furthermore, similarities exist between mathematics research and student learning of mathematics (Cuoco, 2001; Cuoco, Goldenberg, & Mark, 1996; Ernest, 1998; Dreyfus, 1991; Dubinsky, 1991). The existence of these similarities raises the question of whether experience with one process (mathematics research) impacts conceptions of the other (student learning). An investigation of that question would also offer information on the relationship between school mathematics and research mathematics, an issue at the heart of the debate over the value of mathematics immersion experiences for teachers (Proulx & Bednarz, 2009). For these reasons, the question of how mathematics research experiences impact teacher conceptions of student learning forms the core of the study described below. The following few sections will specify the research questions, briefly describe

the methodology used to address those questions, and provide a brief overview of the motivation, both personal and in the literature, for this work. The background and plan for research will be treated in a much fuller and more detailed manner in the conceptual framework and literature review (chapter II) and methodology (chapter III) chapters to follow.

### **Research Questions**

This research addressed one foundational question: **How does a mathematics research experience impact teachers' conceptions of student learning?** Note that this includes not only *what* impact (if any) such an experience has, but also *how* it develops. That is, while I certainly focused on measuring change in conceptions of student learning as teachers participated in a mathematics immersion experience, the study was also designed to gain insight into the ways teachers internalized the experience and related it to their own students' learning. By investigating the *how* in addition to the *what*, the study highlighted parallels teachers drew between school and research mathematics, as well as how teachers used their personal learning experiences as they considered student learning. To focus on these more nuanced aspects of how conceptions of student learning were impacted, I framed three topical sub-questions. The investigation of the sub-questions helped develop a more robust picture of teacher conceptions of student learning in relation to mathematics research experience:

- How does such an experience impact the mathematical learning goals that teachers form for their students; their expectations of student capability, work, and achievement; and their criteria for a successful lesson?

- How do teachers' personal experiences of coming to know mathematics shape their conceptions of student learning of mathematics?
- In what ways, if at all, do teachers construct parallels between the process of mathematics research and student learning? Specifically, how do teachers make connections between the work of mathematicians and that of their own students and between the "advanced" mathematics they explore and the mathematics their students are engaged in?

The first question addresses specific features of teachers' conceptions that were investigated. Namely, it concerns goals and expectations for students – whether, as the teachers reflected on student learning in light of their experience, they began to see students as more or less capable mathematical agents and whether their criteria for a successful lesson changed. A feature of many mathematics immersion or research experiences has been to encourage participants to explore mathematics (Stevens, et al., 2008). Furthermore, for some participants, the ability to control inquiry has been a transformative aspect of the experience and many were surprised by their own capabilities (Marshall, 2008). One might conjecture that this personal empowerment could manifest itself in practice through altered expectations. On the other hand, at least one participant in an immersion experience conceived of mathematics as tidy, ordered, and neat, and found the challenge to that conception to be unpleasant and disconcerting (Badertscher, 2007). A teacher in this situation might establish learning goals in accordance with a value of preventing similar discomfort for students. These outcomes are, of course, hypothetical, but they highlight the



importance of considering these aspects of teacher conceptions of student learning that are closely tied to practice.

The hypothetical scenarios presented above highlight the main thrust of the second question: the ways teachers' personal learning experiences shape the ways they think about student learning. When teachers learn through a given activity, they often attribute the learning as a property of that activity (Heinz, Kinzel, Simon, & Tzur, 2000), indicative of the way teachers make use of their own learning experiences in shaping learning situations for students. Indeed, teachers will sometimes project their own understandings onto students (see, for example, Tzur, Simon, Heinz, & Kinzel, 2001) and explain or predict student difficulties using their own experiences (Badertscher, 2007). Thus, I sought to understand how teachers were drawing on their own research experiences as they considered the learning of students.

The third topical subquestion concerns the relationship between school mathematics (in which the teacher participants were well-versed) and research mathematics (which they experienced during a mathematics immersion program). Though there are similarities between the processes (both social and cognitive) involved in each domain, the question of whether teachers recognize or consider these useful remains open. That is, a teacher may see relationships between the work of researchers and the work of their students, leading to an alteration in the way that teacher thinks about student learning. It may be the case, however, that a teacher sees the domains of research and school learning as disjoint enterprises, and so experience with the former has no bearing on

conceptions of the latter. By seeing how teachers drew parallels (or failed to do so), I was able to explore the relationship between school mathematics and the mathematics of mathematicians, an important task for discussing the nature of school mathematics and an open area of inquiry in the mathematics education community (Proulx & Bednarz, 2009; Watson, 2008; Mendick, 2008; Zazkis, 2008).

### **Brief Overview of Methodology**

I sought to answer these research questions by interacting with teachers before, during, and after their participation in a professional development experience based in providing them experience with mathematics research. I will refer to this particular program as the RLE program<sup>2</sup>, though this is not, of course, the real name. Teachers enroll in the program in question during two consecutive summers. During the first summer, they take a course in number theory and engage in extensive group problem-solving sessions supported by program staff, and the nature of mathematics research is emphasized. They also stress the fact that the problem sets completed by the participants are meant to simulate the research process by starting with exploration of multiple problems in order generate conjectures. Teachers in their second summer take on a research project, and, guided by a faculty mentor, teachers investigate open-ended problems in a manner consistent with the mathematics research process. While the results are not new to the field at large, they do investigate topics that

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<sup>2</sup> RLE for “research-like experience.

are unfamiliar to them as learners, and they are building the knowledge and cognitive schemes for themselves in a manner similar to the way a mathematician might.

The data collection took two forms. First, to provide baseline information about changes in conceptions of student learning, I administered a Likert-scale belief inventory to all willing participants in the program at the beginning and end of the 2009 summer session. This was designed to highlight their beliefs as a group, provide snapshot information about conceptions before and after the experience, and inform the more in-depth case studies of individual teachers. While this instrument measured the existence of impact and provided some information about the nature of that impact, it was too limited to fully accomplish the goal of describing how teachers' conceptions were impacted. To this end, I conducted a series of three task-based, clinical interviews with a smaller group of five teachers. The tasks consisted of lesson planning activities designed to lead teachers to construct *hypothetical learning trajectories* (Simon, 1995; Simon & Tzur, 2004) for their students. By structuring interviews in this way, teachers' conceptions of student learning were encouraged to emerge as they were applied in lesson development. Teachers were asked to verbalize their reasons for making activity choices, their expectations for student responses, and the paths along which they expected student learning to proceed.

In order to develop a picture of the participants' initial conceptions of student learning, I conducted the first interview during the opening week of the summer program. During that interview, I asked teachers to construct and

organize two lessons on familiar algebra topics, and their plans were recorded. After observing the participants as they worked through their projects over the summer, I then conducted another interview in the final week. During this second interview, I presented the teachers with the lessons they developed initially and asked them to make any revisions they might like to and to again construct hypothetical learning trajectories for students in the revised or unrevised lesson. Finally, in the fall, I observed each participant as they taught in their classrooms, and a follow-up interview with each teacher again focused on expectations and conceptions of student learning. The series of three interviews provided information on how conceptions of student learning evolved over the course of the experience and the transition back into the classroom. Following up with teachers in their classrooms highlighted the classroom implications of mathematics research participation by teachers and showed how resilient any changes in teacher conceptions truly were.

## **Background of the Research**

### **Personal Background**

Despite researchers' strive for objectivity, research is not value-free (Glesne & Peshkin, 1992), and "the perspectives and subjective lenses that the researcher and research participants bring to a study are part of the context for the findings" (Schram, 2006, p. 9). Some subjectivity is a necessary and important aspect of this type of research. Therefore, in the interest of full disclosure, I will briefly summarize my personal experiences with coming to know mathematics and the experiences that led to my interest in this particular

research topic. By doing so, I hope to be as forthright as possible about the lens through which this research is viewed.

I come to this research having found exploration in mathematics valuable for my own learning and having used it as a guiding principle in constructing teacher education courses. As a young student, I did well in mathematics mostly because I became very adept at recognizing what problem “type” was being presented and recalling the solution algorithm that applied. In high school, I started to enjoy mathematics as a sort of puzzle, embracing opportunities to experiment and investigate. While many of my classmates dreaded geometry and trigonometric identity proofs, I relished the challenge of finding a route from assumptions to conclusion. My enjoyment of mathematics led me to major in the subject in college, followed by a master’s degree. Through all of these experiences, in addition to my mathematics work as part of the PhD program, I have learned best when presented with a question and left to explore, construct and prove on my own or with a small group of cohorts. Because of that personal experience, I value learning that is offered by exploring open-ended questions individually or with a few others and appreciate the inexactness of much mathematical work as an important aspect of the subject.

After completing my master’s degree, I taught for three years at small, rural, public, liberal-arts college. In this capacity, I helped organize and teach a variety of professional development courses and institutes for in-service teachers. For several of these, we tried to encourage teachers to explore open-ended problems drawn from the subjects they taught or to participate in

investigative activities that encouraged the development of new understandings of familiar concepts. Many participants reported finding the experience valuable for their own knowledge and their classroom practice, which further piqued my interest in the value of exploratory professional development models.

As a PhD student at UNH, I assisted with the evaluation of an NSF grant funding the Center for the Scholarship of School Mathematics (CSSM) at Educational Development Center, Inc. (EDC). The principal investigators of this grant had developed a course for PhD students in mathematics education (see Chazan, et al., 2007) in which participants explored problems motivated by high school subject matter and completed individual projects that simulated mathematics research. The grant brought university mathematics education faculty members from around the country together for a weeklong summer institute with the purpose of modeling the course on a smaller scale with these professionals and encouraging them to develop similar courses at their institutions. Reading about (in participant journals) and seeing the excitement that many of the participants felt just engaging with mathematics again was quite interesting, and the mathematical empowerment many of them described led me to consider in more detail the value of such experiences. It was not a uniformly positive experience, however. A few participants wrote that they used similar models in their own teaching and resented being treated as a student - they did not enjoy being put in a position of such uncertainty.

All of these experiences, especially observing the mathematics educators at the CSSM institute, piqued my interest in mathematics immersion as a

professional development model. While many of the teachers and professionals I encountered reported growing in ways that were consistent with my personal mathematics experiences, this was not uniformly true. Thus, I became interested in how participants experienced these “genuine” mathematical projects and how the experience impacted them. To that end, I was struck by a passage written by Dreyfus (1991) in a chapter on the processes of advanced mathematical thinking:

[There are] very important similarities between the learning process and the research process; namely that in both cases the individual has to mentally manipulate, investigate, and find out about objects, about which his knowledge is very partial and fragmented. Thus, just as the research process is extraordinarily complex, so is the corresponding learning process. It contains the gist of what advanced mathematical thinking is all about” (p. 30)

Subsequent reading indicated further parallels between the processes of mathematics research and student learning in mathematics<sup>3</sup>. That led me to question how experience with the research process may impact a teachers’ conceptions of student learning, ultimately resulting in the questions described above.

Thus, I bring to the research a personal history of finding mathematical exploration valuable for my own learning, and having seen it both empower and discourage educators. My interest in the research is motivated by that history, but not determined by it. My experience has led me to believe that participating in mathematics research (or a simulation thereof) can be a powerful experience, and that it is valuable for certain individuals in certain ways. I cannot, however,

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<sup>3</sup> This literature is reviewed in Chapter II (Conceptual Framework and Literature Review).

assume it to be beneficial in all circumstances. I agree with Proulx and Bednarz (2009) that further research is necessary to understand how educators are impacted by these experiences, which is the reason I decided to undertake the project described below. Of course, my personal interest is hardly justification for undertaking a research project. In the next section, I will show how this research fits into the larger body of mathematics education research and demonstrate how it can help fill an important gap in the existing literature.

### **Rationale and Justification**

In an article in *For the Learning of Mathematics*, Watson (2008) discussed the differences between school mathematics and research mathematics, characterizing them as different domains while still noting the value and importance of encouraging students to behave like mathematicians. Watson's article was followed by several response articles with very different reactions to her premise. Reactions included a focus on the intersections and connections between school and research mathematics (Zazkis, 2008; Zack, 2008); a conclusion that some institutional settings may allow students to "act like mathematicians" (Henderson, 2008); focus on the teacher as the model learner (Povey, 2008); and agreement with the separation, but not with the desirability of students behaving like mathematicians (Mendick, 2008). The variety of responses to Watson's article points out that the intersections between school and research mathematics are still being outlined, and the desirability of increasing or decreasing that overlap remains undetermined. Furthermore, the



disagreement highlights the need for research work that focuses on these open questions.

In a response to Watson's article, Zazkis (2008) commented on the value of having teachers immersed in genuine mathematical practices - some facsimile of the work of a mathematician - in order to make them fully aware of what those practices look like and encourage them to try and make sense of the similarities and differences between school and research mathematics. Indeed, many teacher educators have responded to perceived disconnects between school and research mathematics by encouraging teachers to participate in such "mathematics immersion" experiences. The National Research Council (1999) recommends providing all undergraduates, especially teachers, experience with mathematics (and other disciplines) as it is performed by professionals in the field. Cuoco (2001, quoted earlier), based on his experience as a teacher, teacher educator, and curriculum developer, recommends research experience for teachers as one of the most formative and valuable experiences a teacher can have. A variety of programs designed to develop proficient teaching have adopted this approach, and are supported by recommendations that teachers must increase their base of knowledge of mathematical content and about mathematics as a discipline (National Research Council, 2001). The Conference Board of the Mathematical Sciences (CBMS, 2001) phrases such a recommendation for prospective teachers as follows:

Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching. Mathematics is not only about numbers and shapes, but

also about patterns of all types. In searching for patterns, mathematical thinkers [ . . . ] take actions like representing, experimenting, modeling, classifying, visualizing, computing, and proving. Teachers need to learn to ask good mathematical questions, as well as find solutions, and to look at problems from multiple points of view. Most of all, prospective teachers need to learn how to learn mathematics. (p. 8)

The report goes on to reiterate the need for high school teachers, in particular, to experience and be proficient with the patterns of thought characteristic of mathematicians. Similar recommendations have been made in *Everybody Counts* (National Research Council, 1989), *A Call for Change* (MAA, 1991), and NCTM's *Professional Standards for Teaching Mathematics* (NCTM, 1991). Though these recommendations indicate that a mathematics immersion experience might be important for teachers, Proulx and Bednarz (2009) warn against assuming uniformly beneficial results from such programs, citing a lack of evidence for that claim and encouraging further research to refine, understand, and justify their implementation. I propose that this research project contributes to just such an effort.

Moreover, mathematics education reform efforts, based on extensive research, encourage allowing students to conjecture, construct, and explore in mathematics (National Research Council, 2001; NCTM, 2000), combating the tendency to emphasize the product of mathematical thought rather than the process of mathematical thinking in the classroom (Stigler & Hiebert, 1999; Tall, 1991; Skemp, 1987). Several curriculum design efforts have arisen to support this by using an emphasis on the process of mathematical thinking as a guiding philosophy for development (Cuoco, Goldenberg, & Mark, 1996; NCTM, 1989).

However, instructional materials alone are not enough to instigate the desired change (Cooney, 1988), and research indicates that teachers' conceptions impact the ways they implement curriculum (Lloyd & Wilson, 1998). Indeed, research indicates that teachers do not use reform-oriented curriculum materials as intended when their underlying beliefs are inconsistent with the foundational philosophies of the curriculum (Philipp, 2007; Remillard & Bryans, 2004; Romberg, 1997).

Thus, in order for students to experience "genuine mathematics" (the product and process of mathematics), it is necessary for teachers to have experience with the processes of mathematical thinking they will be expected to teach. Such experience is recommended in general as an important form of teacher education. However, we must not assume that simply providing a "genuine mathematical" experience is enough to equip teachers as envisioned in policy documents and by curriculum developers. This research project investigated how participating in a research simulation impacted teachers' conceptions of student learning. By investigating that question, the research assessed the value of mathematical immersion experiences. Furthermore, participants, as teachers "researching" mathematics, sat squarely in the intersection of school and research mathematics, and outlining the ways in they drew parallels will provide information about the relationships between these two domains, a matter of much debate. The ways in which teachers' personal learning experiences shape their professional work helped illuminate the effectiveness of professional development models based on educating teachers

about subject matter in order to impact the ways in which they teach. Moreover, by focusing on teacher conceptions, this work can shed light on how teachers use and shift their conceptions as they make sense of learning experiences. I turn now to describing in detail the conceptual lens through which I view the work, with an emphasis on the reasons behind these choices. I will also situate it within the relevant literature in order to provide a context for this project.

## CHAPTER II

### CONCEPTUAL FRAMEWORK AND LITERATURE REVIEW

#### Introduction

##### Choosing a Lens

The adoption of a particular perspective entails more than just a set of background opinions that frame the research - the implications include choices of methodology as well as the research questions themselves (Cobb, 2007). Indeed, the conceptual models that underlie a researcher's work are the starting point for that work, providing context and guiding its development (Fawcett, 1999). In the sections that follow, I will describe the theoretical framework for this proposal and highlight those research findings that provide the research context for it. In doing so, I will illuminate the role these things played in formulating this particular research. First, perspectives on the nature of mathematics and mathematics research will be discussed, both as they have appeared in the research literature at large and as I conceived of them for the purpose of this project. I will underscore how an understanding of these contributes to an overall epistemology. The nature of student learning in general, with specific emphasis on mathematics learning, will then be discussed, and the viewpoint from which I undertook this research will again be highlighted. In doing so, the similarities and differences between mathematics research and student

learning of mathematics will emerge and be noted. I will also spend some space clarifying the nature of and relationships between beliefs, knowledge, and conceptions, defining these terms as they were used in this research project. Finally, research related to the content and impact of teacher conceptions will be reviewed in order to set a research context for this study. At all points, I will attempt to make clear the reasons for the choices I have made in order to make explicit the lenses through which the work is being viewed and the means through which its development and interpretation took shape.

### **Mathematics and Mathematics Research**

#### **Perspective on the Nature of Mathematics**

This study will approach mathematics from what Ernest (1998a) terms a *social constructivist* perspective, which acknowledges that mathematics is a human activity and means of engagement rather than simply a body of knowledge (Zazkis, 2008). Drawing on Wittgenstein's notion of language games (see Wittgenstein, 1978) and Lakatos' logic of mathematical discovery (see Lakatos, 1976), this perspective considers mathematical knowledge as subjective, but socially constructed. As such, the view of mathematical concepts takes into account both the individual creativity that is brought to bear in the construction of these concepts and the social milieu in which that construction takes place. As Ernest (1998a) puts it:

The social constructivist account of the objects of mathematics is thus a combination . . . of (1) mathematical imagination and intuition which emerges from the human capability to construct (in stages) and hence to recall or retrieve imagined worlds . . . and (2) human cultural, discursive signifying practices, which, having been individually appropriated, provide the resources for (1). (p. 219-20)

Ernest (1998a) goes on to describe how Wittgenstein repurposed the philosophy mathematics to be descriptive and observational of mathematics rather than an attempt to validate it. I argue that this is an important perspective for this research which is not concerned with making sure mathematics is philosophically or logically valid, but only with teachers' interactions with it and the impact that interaction has on them. That is, this project is concerned with how mathematics is investigated, performed, and created by people.

Viewing mathematics as a human activity necessitates consideration of the humans engaged in it. To highlight the social, yet subjective nature of mathematical activity, the discussion below focuses on two distinct yet interrelated groups of mathematical participants. First, the nature of mathematics research will be explored through the reflections of researchers themselves in order to gain insight into the activity of mathematical investigation and creation. Secondly, the nature of student learning of mathematics will be discussed. In both cases, I will show how the social constructivist philosophy of mathematics is related to knowledge development by discussing the jointly individual and social nature of each. Parallels between the processes of mathematics research and student learning of mathematics will arise, with implications for the mathematical immersion of teachers.

### **Perspective on the Nature of Mathematics Research**

#### **The Culture in Which Mathematics Research Takes Place**

Thurston (1994) wrote that “mathematicians apparently don’t generally rely on the formal rules of deduction as they are thinking” (p. 164), and Ervynck

(1991) indicated that the creation of new mathematics seems to take place outside the rigid formal structures of mathematics. Thus, even researchers themselves, comfortable with the formalities of mathematics, recognize the human activity that is present but separate from those formalities. The discussion of mathematics research found below will focus specifically on the activity of researchers in mathematics, with special attention paid to the informal (in the sense of Thurston's quote above) reasoning processes through which they operate. Through this description, I will further explicate the notion of "mathematical immersion" for teachers or learners.

Davis and Hersh (1981) made the point that a large gap exists between "the actual work and activity of the mathematician and his own perception of [it]" (p. 34). This quote points out a fundamental problem in discussing the mathematics research process – that is, mathematical communication typically focuses on results rather than the process by which those results are obtained (Thurston, 1994; Muir, 1996). In fact, an emphasis on the product of mathematical thought over the process of mathematical thinking has extended into the communication of mathematics in the classroom (Dreyfus, 1991; Skemp, 1987; Tall 1991). Mathematician Allan Muir (1996) vividly described this phenomenon as he somewhat hesitantly divulged the nature of his own mathematical activity:

Unless you're very clear about how to proceed with a problem, which probably only occurs when an attempt is not truly exploratory . . . you struggle up innumerable blind alleys making all sorts of false starts, mistakes, reworkings and sudden changes of direction. Maybe you do arrive somewhere in the end, but anyway you report to others a polished version which disguises the struggle. Indeed,



such is the ethos prevailing within mathematics, that we find it positively embarrassing to reveal how stupid we feel ourselves to be for not seeing our final conclusions from the outset. (p. 1)

William Thurston (1994) made striking note of how problematic such a culture can be by contrasting the detrimental impact of his early work, when he did not share the reasoning that led him to the results, on others' excitement about the field with the flourishing of ideas resulting from a later effort to share his mental models.

Taken together, these anecdotes highlight the importance of communicating the process of mathematics research. A persistent failure to do so has led to an abandonment of the mathematical field by many students in a manner similar to that seen by Thurston as a result of his early work. Pulling back the curtains that hide the true nature of mathematical work is a key aspect of a mathematics immersion experience (Mendick, 2008). To quote Muir (1996) again:

We [mathematicians] normally disguise from others, and perhaps even from ourselves, that the majority of mathematical work is a struggle through uncharted bog, most of which peters out in boredom or disillusion. (Again I feel a collapse of confidence – do I speak only for myself here?) If this is indeed the major part of mathematics as a process, then our efforts to understand that process should focus, in proportional measure, on all those things which get left out from the final presentations – errors, misunderstandings and the like; and not just these but also social and inter-personal motivations for one's work. (p. 1)

The struggle that goes along with the reasoning behind results remains hidden and often undiscussed. Mathematics immersion programs tend to agree with Muir that efforts to understand mathematics as a process must focus on the messy parts. I transition to a discussion of what those messy parts look like with

two quotes from important and primary sources - mathematicians writing about their personal experience with mathematics as a process. Mathematician and philosopher Gian-Carlo Rota wrote:

[A] mathematician's job is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks. (Davis & Hersh, 1981, p. xviii)

In an interview on the PBS program *Nova*, Andrew Wiles, famous for proving Fermat's Last Theorem, described the navigation of Rota's "tangle" with a vivid metaphor:

Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of- and couldn't exist without – the many months of stumbling around in the dark that precede them. (quoted in Byers, 2007, p. 1)

In order to investigate the activity of mathematics, it will be necessary to focus on Rota's "tangle" and Muir's "uncharted bog", on how mathematicians negotiate Wiles' darkened mansion - the various ways in which mathematics research is undertaken and mathematical knowledge is created.

### Intuition and the Role of the Individual

In his classical investigation of mathematical invention, Hadamard (1945), building on Poincare's 1908 lecture on the subject (see Halsted, 1946) and the accounts of mathematicians, outlined stages of creative work. First, there is a

period of intense preliminary work (perhaps algorithmic, perhaps familiarization with results and ways of operating developed by others) that is fully conscious. Ervynck (1991) also conjectured that, through action, mathematical processes are interiorized at the outset of mathematical creativity, thus becoming useful as mental mathematical objects. The preliminary work is followed by a period of incubation during which the mathematician steps away from the problem, leading to a flash of insight that he terms “illumination preceded by incubation” (p. 35). Though the length and nature of the period of incubation seems to differ among various researchers<sup>4</sup>, the flash of insight is a common experience (Burton, 1999).

Navigation of the creative process has also been described as movement through multiple layers of horizontal and vertical mathematizing (Rasmussen, et al, 2005). Horizontal mathematizing “refers to formulating a problem situation in such a way that it is amenable to further mathematical analysis” (ibid, p. 54) - the preparatory work described by Hadamard and Ervynck, the making of connections between existing knowledge, the sniffing of patterns and experimentation that inspires conjectures or provides means of attack (Cuoco, Goldenberg, & Mark, 1996). Such horizontal mathematizing serves as the foundation for vertical mathematizing, which is the creation of new mathematical realities (Rasmussen, et al, 2005), leading in turn to further horizontal and vertical mathematizing. A great many research mathematicians have

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<sup>4</sup> Poincaré reported insight as being preceded by a sleepless night, while others reported it coming when fatigued from long periods of conscious work (in the case of Sterling), or after this fatigue has passed (in the case of Helmholtz) (Hadamard, 1945).

characterized their work as making connections in both the horizontal and vertical sense (Burton, 1999).

The characterizations of mathematical activity described in the preceding paragraphs focus heavily on the subjective experience of the individual, with an emphasis on a mathematician's cognitive work. Echoing Hadamard's emphasis on the important role of individual human intuition and informal reasoning in mathematics research, Thurston (1994) noted:

[P]ersonally, I put a lot of effort into 'listening' to my intuitions and associations, and building them into metaphors and connections. This involves a kind of simultaneous quieting and focusing of my mind. Words, logic, and detailed pictures rattling around can inhibit intuitions and associations. (p. 165)

Taken together with Muir's description of his own reasoning and the various ways of reasoning described by Hadamard's (1945) respondents, this quote highlights the idiosyncratic ways of thinking and assortment of conceptual forms developed by individuals through their informal reasoning. Each mathematical agent seems to be constructing images, phrases, rhythms, words, or metaphors (Lakoff & Núñez, 2000; Thurston, 1994) that somehow embody the concepts in a personally relevant way. Even researchers working in the same field might understand the same concept in different ways (Byers, 2007). Mathematical historian Morris Kline (1980) summed up that idea by writing that:

mathematics creates by insight and intuition. Logic then sanctions the conquests of intuition<sup>5</sup> [. . .] the whole structure rests fundamentally on uncertain ground, the intuition of humans. Here

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<sup>5</sup> The quote "Logic then sanctions the conquests of intuition" is attributed to Jacques Hadamard.

and there an intuition is scooped out and replaced by a firmly built pillar of thought; however, this pillar is based on some deeper, perhaps less clearly defined, intuition. (p. 408)

### The Role of Social Influences and Communities of Practice

While intuition and individual ways of thinking are certainly important aspects of mathematics research, it must be noted that mathematics research is far from an entirely individual endeavor. In fact, at least one study (Burton, 1999) found that the majority of research mathematicians don't see mathematics as an individual activity. Almost all the participants in the study (66 out of 70)<sup>6</sup> claimed to do at least some collaborative work, and even those who felt research was a strongly individual activity acknowledged "at the same time, there is a common pool of ideas which is the driving force to what is going on" (p. 127). In fact, there is a powerful social component to mathematics research, and an individualistic approach to studying it will provide an insufficient description (Muir, 1988).

As individuals engage in the enterprise of mathematics research, their interactions with each other and with the world assemble into a shared vision of the goals and norms of the enterprise in a way typical of a *community of practice* (Wenger 1998), which Wenger described as follows:

Over time, [. . .] collective learning results in practices that reflect both the pursuit of our enterprises and the attendant social relations. These practices are thus the property of a kind of

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<sup>6</sup> The participants in the study, split evenly between males and females, were research mathematicians of varying experience in an assortment of specialties, including statistics, applied mathematics, and theoretical mathematics. They were interviewed individually in order to gain insight into the practices of mathematicians, with an eye toward the implications these practices might have for coming to know mathematics in general.

community created over time by the sustained pursuit of a shared enterprise. It makes sense, therefore, to call these kinds of communities *communities of practice* (p. 45, italics in original)

Participation in a community of practice is a powerful force for learning, and initiation into one has implications for how that learning might proceed, and it is therefore an important aspect of mathematics immersion programs just as it is for mathematics research. In mathematics research, social practices often give directions for investigation or hints on how to proceed (Boaler, 2002). Over years, the community of practice has determined what forms of mathematical communication are acceptable and the how validity is established (Hanna, 1991; Weber, 2008). The community determines the worth and utility of a particular work and what open questions are or are not important (Davis & Hersh, 1981; Thurston, 1994). The questions that are investigated and the ways in which they are approached are often influenced by a desire to gain “credits”<sup>7</sup> in the community of practice (Thurston, 1994) or because of a desire to help or impress colleagues (Muir, 1996).

Lakatos’ (1976) reconstruction of the social negotiation of Euler’s conjecture<sup>8</sup> shows the social nature of mathematical creation; that mathematics, through conjecture, proof, and criticism, grows as a succession of improved guesses. For this reason, mathematics can be considered as a conversational

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<sup>7</sup> Thurston characterizes “credits” as those activities that increase a mathematician’s stature in the community. These include publications or results the community determines are valuable, unique, impressive, or important.

<sup>8</sup> That is, for all polyhedra of a certain type,  $V-E+F=2$ , where  $V$  is the number of vertices,  $E$  is the number of edges, and  $F$  is the number of faces.

discipline (Ernest, 1998b). Individuals contribute their insight to this conversation, but the discipline grows through the work of the community, and the practices of that community shape and contribute to the individual insights. Thus, though mathematics research is created in part by individuals, that work is subject to a powerful socio-cultural situatedness. It is neither wholly individual nor wholly social. Thus, mathematics research activity, a subset of the mathematical field, is integrated into the social constructivist philosophy of mathematics.

### Reasoning Processes in Mathematics Research

Because of the subjective nature of mathematical knowledge, several researchers have attempted to characterize the means by which individuals construct mathematical knowledge, or, more generally, reason mathematically. Such means of reasoning have been variously described as mathematical ways of thinking (Harel & Sowder, 2005) or *habits of mind* (Cuoco, Goldenberg, & Mark, 1996). *Ways of thinking* are those reasoning practices that are broadly applicable rather than situated in a particular problem or context, governing in a general way the approaches one takes to understanding specific mathematical situations. They “involve at least three interrelated categories: beliefs, problem-solving approaches, and proof schemes” (Harel & Sowder, p. 31). *Habits of mind* are defined similarly, as general approaches to mathematical work that “have shown themselves worthwhile over the years” (Cuoco, Goldenberg, & Mark, 1996, p. 376).

An exhaustive list of the ways of thinking and habits of mind employed by mathematicians is unrealistic and probably pushes the boundaries of being useful. A more realistic goal is to broadly describe some of the cognitive processes used in mathematics research, and that is the aim here. For example, Tommy Dreyfus (1991) categorized the processes of advanced mathematical thinking<sup>9</sup> into two broad types: *representing* and *abstracting*. *Representing*, whether formal or informal, allows one to capture the meaning of a concept in its representation. The processes involved include visualizations such as those discussed by the subjects of Hadamard's study, symbolizing, switching representations, translating between representations, and modeling. Cuoco, Goldenberg, and Mark (1996), listed various habits of mind, including those they view as desirable in students and those they consider to be present in mathematicians. They use different labels and categories, but highlight several that are related to Dreyfus' "representing" group. For instance, they claim students should be "describers" and "visualizers", that mathematicians "use abstraction" and "represent things" in addition to "talking big and thinking small" (that is, they talk in generalities but think in terms of specific instances) and "us[ing] a common language" while sometimes making use of multiple languages and multiple points of view. Thus, the various processes by which representation occur are important aspects of the mathematical process as a whole.

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<sup>9</sup> A discussion of the various meanings attributed to this term is beyond the scope of this work, but for the purposes at hand, it may sufficiently be described as the thinking necessary for doing mathematical work above the secondary level, certainly including mathematics research.



In Dreyfus' (1991) description, *abstraction* is used to mean a mental reorganization of schemes, "building mental structures from . . . properties of and relationships between mathematical objects" (p.37). He characterized it as the fundamental mathematical learning (and therefore, creating) process. It includes generalizing and synthesizing. In this vein, Cuoco, Goldenberg, and Mark (1996) claimed students should be "pattern-sniffers" and "inventors", and that mathematicians, "talk small and think big" (that is, they try to generalize local results), "extend things", and "push the language" to see if patterns of notation and representation can be extended.

Harel and Sowder (2005) discussed "ways of thinking" more concretely as beliefs about mathematics, problem-solving approaches, and proof schemes. That is, rather than thinking about them as processes, they conceive of them as a set of available actions and schemes situated in useful frames of mind. Cuoco, Goldenberg, and Mark (1996) similarly outlined some additional habits of mind that are more action-oriented, encouraging students who are "experimenters", "tinkerers", "conjecturers", and "guessers". They claimed mathematicians "mix deduction and experiment", notice change or the absence of it, and make use of calculations and algorithms (ibid, 1996).

Ways of thinking and habits of mind are a useful construct for considering the processes involved in mathematical reasoning and creation as well as the actions and mindsets that have proven useful to horizontal and vertical mathematizing in mathematics research. Cuoco, Goldenberg, and Mark (1996), proposed that many of the habits of mind used by mathematicians are useful for

mathematics learners at all levels. Indeed, many of the aspects of mathematics research described above have parallels in student mathematical learning. The following section will consider the nature of student learning in mathematics, highlighting the ways in which that process and the process of mathematics research are both similar and different in order to frame the discussion of the interaction between experience with research and teacher conceptions of student learning.

### **The Nature of Student Learning of Mathematics**

#### **Choosing a Lens**

Investigating questions of what it means to learn or to know involves seeking indirect evidence and drawing conclusions based on oblique data. Thus, any perspective on what it means to learn or to know is bound to provide only an incomplete description, so even contradictory philosophies may all have valuable explanatory power. The purpose of choosing an epistemological perspective, then, is not to dictate a set of truths, but rather to provide a lens that provides meaning and context for observed phenomena (Cobb, 2007). For this project, I view knowledge development in a way similar to and consistent with the view of mathematics espoused above, as jointly individual and social in nature. That is, I take the point of view that the individual ultimately constructs knowledge, but is affected by context in ways that are inseparable from the act of construction. Thus, both radical constructivist and sociocultural philosophies offer valuable insights into learning, but both also fail to account for some aspects of that process (Cobb & Yackel, 1996). Below, I will discuss the contributions both

make to a fuller understanding of mathematics learning. Moreover, I will highlight the shortcomings of considering one perspective in isolation. Finally, aspects of the two extremes will be merged to underscore the *social constructivist* perspective on knowledge construction that has shaped the design, implementation, and interpretation of the research at hand.

### **Subjective, Psychological Construction of Knowledge**

Constructivist epistemology is generally based upon the relatively simple idea that individuals construct their own knowledge, an idea supported by all the “in-the-head” processes of mathematics learning described in the previous section. The term *constructivist*, however, has often been appropriated for whatever use a particular researcher or author chooses<sup>10</sup>, and perspectives so labeled often intermingle and “shape-shift”, taking on slightly different meanings and being used for slightly different purposes depending on the context (Oxford, 1997). However, generally it means that learning occurs as individuals, reflecting on their own activity and experience, reorganize their psychological schemes for making sense of the world accordingly (von Glasersfeld, 1989, 1990). Thus, discussing the body of mathematical knowledge is meaningful only as it pertains to the knowledge held by individuals. From this perspective, the existence, truth, and validity of mathematical objects or processes is subjective.

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<sup>10</sup> These still typically share a common value of knowledge organization and development being subjective to the learner and a common source in the work of Piaget (1970)

Taking such a position provides some insight into mathematics learning because focusing on the cognitive structures and processes of individuals can illuminate how knowledge is held and developed. Research indicates that the metaphors and analogies people use to reason mathematically are often idiosyncratic (English, 1999; Lakoff & Núñez, 2000) and that understandings often vary from individual to individual. Erlwanger (1973) famously illustrated this in his case study of Benny, a sixth-grade student who adapted his unique fraction schemes to replicate patterns in a manner that allowed him to achieve satisfactory progress on his computer-based individualized mathematics program. Benny arrived at a set of procedures that he was confident and consistent in using, and those procedures made sense with respect to the criteria he had set for them in spite of a lack of mathematical sense (*ibid*, 1973). Benny's case illustrated that some account of the ways in which individuals self-organize knowledge is necessary, and a constructivist perspective provides this (*cf.*, Piaget, 1970; von Glasersfeld, 1989). In addition, it helps explain teacher-to-teacher differences in knowledge and practice, thereby providing a richer account of how individual teachers shape their classrooms (Putnam & Borko, 2000). Therefore, a constructivist perspective on learning, by focusing on individual cognition, "can be very powerful in helping to study mathematical learning, to develop appropriate teaching strategies, and to reflect on the everyday problems of schoolteaching" (Noddings, 1990, p.194).

On the other hand, there are major aspects of learning and knowledge construction that a completely subject viewpoint fails to capture. For one, it does

not account for social influences on individual cognition. The fact that the same conception is often held by many individual across various contexts (intersubjectivity) is evidence for the existence of such influences (Kirshner & Whitson, 1998; Lerman, 1996). Furthermore, issues arise when considering social activity and communal learning. As described above, mathematics researchers are participating in a community of practice (Wenger, 1998), and what is regarded as acceptable reasoning or justification is socially determined by the community. In much the same way, the individual construction of knowledge is shaped by the social constraints placed on it. Benny's fraction schemes were so inconsistent with standard mathematical practice in part because his context guided him to build them instead of initiating him into standard ways of operating (Erlwanger, 1973). In order to account for this, I will next discuss social influences on and the context-dependent nature of student knowledge development, followed by a discussion of how the two extreme positions might together form a more satisfactory explanatory viewpoint.

### **Social Construction of Knowledge**

Rooted in the work of Vygotsky, a sociocultural perspective stresses that mental functions have their roots in social creation (Forman, 2003). An assumption is made that humans are social beings, and that such an idea is key for understanding how learning occurs – that human social nature is fundamental to how one lives, acts, learns, and interacts with the world. This attention to context highlights the important role an individual's (a student's, in particular) communities of practice play in learning, "and in spite of curriculum, discipline,

and exhortation, the learning that is most personally transformative turns out to be the learning that involves membership in these communities of practice” (Wenger, 1998, p. 6). Vygotsky also recognized that students’ use of tools (including symbols) deeply affects the development of mathematical. From his perspective, all activity is mediated by signs<sup>11</sup>, all of which are given meaning via social negotiation (Forman, 2003). Dörfler (1993) characterized the sociocultural perspective and its relevance for instruction as follows:

[Thinking] is no longer considered to be located exclusively within the human subject. The whole system made up of the subject and the available cognitive tools and aids realizes the thinking process. [. . .] Mathematical thinking for instance not only *uses* those cognitive tools as a separate means but they form a constitutive and systematic part of the thinking process. The cognitive models and symbol systems, the sign systems, are not merely means for expressing a qualitatively distinct and purely mental thinking process. The latter realizes itself and consists in the usage and development of the various cognitive technologies. (p. 164)

The response of the brain to external stimuli suggests a close link between in-the-head cognition and involvement with the external world (Kirshner & Whitson, 1998), so this perspective offers several insights for considering student learning in mathematics. For example, it helps explain how the context in which knowledge develops plays a crucial role in the nature of that knowledge, and how knowledge is often situated in a particular context (Anderson, Reder, & Simon, 1996; Kirshner & Whitson, 1998). School mathematics often does not translate to practical or informal situations calling for the same operations, and mathematics learned in informal settings sometimes fails to transfer to school

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<sup>11</sup> “Signs”, in this case, includes language.

mathematics (Carraher, Carraher, & Schliemann, 1987). In addition, a sociocultural perspective on learning highlights how mathematics is developed through conversation and communication, recognizing that school mathematics is a community activity (Ernest, 1998b; Lakatos, 1976). The fact that classroom culture and teaching practices impact student learning (see Hiebert & Grouws, 2007) is evidence that context and community components are important aspects of learning (Greeno, 2003).

A sociocultural perspective also stresses the important role that socially-negotiated signs and symbols play in mathematics learning and reasoning (Forman, 2003), and that knowledge is fundamentally shaped by the names or representations that are used to embody particular concepts (Pimm, 1995). Studies have indicated that students who were allowed to construct their own symbols developed robust conceptual understandings that are sufficient for further mathematizing (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). Furthermore, other studies have shown that different representations for the same concept yield different understandings (Falcade, Laborde, & Mariotti, 2007), and the representations that children use in problem-solving have consequences for the ways they approach those problems and the understandings that develop from them (Smith, 2003). Thus, signs, symbols, representations play an important role in knowledge and learning, and by attending to these, sociocultural theory contributes valuable insights.

Still, a sociocultural perspective on learning fails to paint the whole picture. It gives short shrift to idiosyncratic knowledge and the differences among

individual experiences of context, community, and social milieu. Furthermore, sociocultural theory makes the argument that internal cognitive processes are all the result of internalizing social practices, but these practices must, in turn, be negotiated by individuals. This devolves into an infinite chicken-and-egg regress that is problematic. Thus, while sociocultural theory lends a tremendous amount of explanatory power to the theory of learning, it, like the theory that knowledge is completely subjective described above, leaves significant holes. In the next section, it will be shown how some reconciliation of these two perspectives gives a fuller picture of learning in mathematics.

### **A Social Constructivist Perspective on Knowledge**

The contributions and shortcomings of the viewpoints described above demonstrate why I strive here for a theory of knowledge that accounts for both individual and social components of knowledge development. However, the two viewpoints take fundamentally different views of the world and an individual's role in it. On one side, there is no knowledge apart from that which an individual experiences or constructs. On the other, knowledge is socially-possessed, and thus individuals internalize, rather than construct, knowledge. Simply declaring the two complementary and proceeding as if both are true leads to incoherence (Lerman, 1996). Thus, it's inappropriate to simply cherry-pick desirable aspects of each perspective while ignoring their contradictory implications. In order to successfully account for the contributions of both individual cognition and social processes to the construction of knowledge, it is necessary to coordinate the two



rather than attempt to classify learners' activities as one or the other (Cobb & Yackel, 1996).

Therefore, the design, implementation, and analysis of this study was undertaken from the perspective that knowledge in general, and mathematics knowledge in particular, is held by social individuals. That is to say, I consider knowledge to be constructed through the coordination of psychological construction and social interaction, discourse, conversation, and other social processes. Knowledge may grow through the myriad ways in which such coordination may occur. For example, individual conceptions may result from internalizing social norms, values, and practices. However, subjects will internalize those aspects of their context in different ways – a shared context experienced subjectively. Furthermore, each of those individuals contributes to the social experience of the whole. Thus, social and cognitive processes each play integral roles in shaping each other and contributing to knowledge development. I refer to this as a *social constructivist* perspective in much the same sense as previous researchers.

Cobb and Yackel (1996) show how such a perspective can be used to describe correlations between social and psychological perspectives. For example, in analyzing individual and collective activity in the classroom, they found it useful to consider classroom mathematical practices and individual mathematical conceptions and activity as social and psychological analogs (ibid, 1996). Note how these two areas necessarily overlap and each contributes to the other - the fullest picture of classroom activity must consider both. So though

it might be appropriate or useful to consider the strictly social or cognitive aspects of knowledge in a given research situation, this must always be done with the recognition that those aspects interact and depend on each other in complex ways.

This choice of theoretical perspective had a variety of implications for this study that point to the pragmatism behind the choice. I investigated the impact that participation in a mathematics immersion experience has on teachers' conceptions of student learning in mathematics, necessitating a framework for discussing both how teachers conceive of student learning and how teachers themselves participate in mathematics. A social constructivist perspective accounting for the coordination of social and cognitive processes in mathematics learning does so. The perspective on learning described above provided an interpretive lens for teachers' hypothetical learning trajectories. As such, it permitted interpretation of teachers' conceptions of both social and psychological aspects of learning. In other words, just as it provides a more fully-realized explanation of learning in general, this framework offers greater range of interpretive possibilities and the potential for richer descriptions of teacher conceptions. Other researchers have found that coordinating social and psychological perspectives allows for explanations of how changes in beliefs can be fostered and nurtured in the classroom (Yackel & Rasmussen, 2002), and it offers the same explanatory power for changes in teachers' beliefs and conceptions.

However, the theoretical perspective served not only as an interpretive lens not only for teachers' conceptions of student learning, but also for their participation in the mathematics immersion experience. As noted above, social norms and influences strongly interact with individual cognition and intuition in mathematics research, and social constructivism can be suitably modified to function as a philosophy of mathematics. I purposely adopted a perspective on student learning in mathematics that is consistent with the experiences of those who do, learn, and create mathematics professionally. By considering the coordination of social and cognitive processes, one can provide an account of both teacher learning and teacher participation in mathematics. Not only is this useful for considering how people learn as they create mathematics in ways similar to professional mathematicians, it also suggests parallels between mathematics work at the upper and lower levels of research and student learning in school. The next section will consider these parallels, as well as those aspects of mathematics research and student learning that are dissimilar, in order to propose how experience with mathematics research might impact teachers' conceptions of student learning in mathematics.

### **Similarities and Differences Between Mathematics Research and Student Learning**

#### **Similarities Between Mathematics Research and Student Learning**

In his classic book *How to Solve It*, Polya (1957) outlined a general heuristic for approaching mathematical problem solving consisting of four basic steps (*understand* the problem, *formulate* a plan, *carry out* your plan, and

*examine* the solution). Principally intended to aid students with mathematical problem solving, Polya's steps also apply as a general heuristic for problem solving at any level of mathematics. Indeed, in his introduction to the first edition, Polya notes that he is making a first attempt to present mathematics "in the process of being invented" (p. vii) to the student. In other words, he is suggesting that the means of mathematical creation are applicable to students learning mathematics. The group of students in Lakatos' (1976) *Proofs and Refutations* modeled the nature of mathematical creation. The author's principal purpose was to establish a philosophy of mathematics, but he nevertheless presented the conjecturing, negotiating, convincing, and connecting that occur as mathematical knowledge is created. The students in Lakatos' book were not developing knowledge that was new to the field, but they were developing knowledge that was new to them, suggesting that this model of knowledge creation is at least partly applicable to student learning in general. In fact, Larson and Zandieh (2007) presented classroom episodes that showed how the discovery methods described by Lakatos could be adapted to construct a framework for considering activity in the mathematics classroom. Their conclusions suggested that Lakatos' notion of guided reinvention can be utilized as a useful heuristic for instructional design.

The existence of general heuristics for mathematics work, such as those described by Polya and Lakatos, indicates that mathematics learning at all levels consists of solving problems - whether they are posed by others, based on conjectures by the learner, or arise to resolve tensions and inconsistencies in

existing knowledge<sup>12</sup>. Thus, contrary to Watson's (2008) suggestion, perhaps research and school mathematics may be conceived as parts of a single whole rather than as two separate processes, so similarities or parallels between the two processes may exist at a variety of levels.

One immediate similarity is the existence of hidden complexities below the surface of both. This was highlighted by Dreyfus (1991) in the quote that initially directed me toward the similarities between the mathematics research and learning processes (see p. 13):

[There are] very important similarities between the learning process and the research process; namely that in both cases the individual has to mentally manipulate, investigate, and find out about objects, about which his knowledge is very partial and fragmented. Thus, just as the research process is extraordinarily complex, so is the corresponding learning process. It contains the gist of what advanced mathematical thinking is all about (p. 30)

Indeed, the framework for mathematics work as horizontal and vertical mathematizing functions equally well as a description of the work of learners and creators of mathematics (Rasmussen, et al, 2005), and the act of making connections, described by mathematicians as a key part of their work, is an important action for mathematics students as well (Boaler, 2003). Furthermore, social processes shape, and are in turn shaped by, individual cognition. That is, student learning cannot be understood as fully individual or fully social in nature, and it is useful to think of it as the coordination both (Cobb & Bauersfeld, 1995;

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<sup>12</sup> This has been characterized as overcoming of epistemological obstacles (Harel & Sowder, 2005; Brousseau, 1983) in the course of learning.

Cobb & Yackel, 1996; Cobb, Yackel, & Wood 1993), just as it is for mathematics research.

As noted earlier, mathematicians typically report only the product of their exploration through Muir's "uncharted bog" or Wiles' "darkened mansion", often disguising the struggle that was necessary to achieve it. That product of an often extensive and laborious journey is then judged by external, socially-negotiated standards of truth. A similar phenomenon occurs in school mathematics. The principally summative assessments of students do not often attend to the thinking behind their work (William, 2007), so students are asked to report only on the product, rather than the process, of their work. That product is then judged according to a standard of truth established by the school, educational, and mathematical communities. There is a parallel between expected justification in demonstrating mathematical knowledge and assessment of mathematical learning (Ernest, 1998b).

Furthermore, many of the cognitive and social processes involved in research and learning in mathematics are the same. When describing the psychological processes involved in advanced mathematical thinking (particularly *representation* and *abstraction*), Dreyfus (1991) noted that many of them "occur at *any level* of mathematical thinking" (p.34, italics added). Piaget (1970) described a specific type of abstraction, *reflective abstraction*, as the coordination of actions to form new mental objects and actions. He conceived it to be the key component of mathematical knowledge formation. In fact, he considered the history of mathematics to be an ongoing process of reflective abstraction (Piaget,

1985). Since it describes aspects of mathematics learning in young children as well as the formation of new advanced mathematical knowledge, reflective abstraction is a fundamental cognitive tool for mathematical work at any level (Dubinsky, 1991). Other similarities exist on the individual psychological level. Mathematical reasoning at both the research and school levels occurs, at least partially, through metaphor, metonymy, analogy, and imagery. Though certain images or metaphors are common, they are often idiosyncratic and personal (Lakoff & Núñez, 2000; Presmeg, 1992). In fact, being equipped with metaphors that are often meaningful to students and understanding how students make use of those metaphors is an important aspect of mathematics for teaching (Presmeg, 2006). Moreover, metaphors have been shown to be important for teachers being enculturated into mathematical practices (Chapman, 1997), indicating that they play a role in mathematical reasoning in general, not just in relation to particular content or concepts. All individuals engaged in mathematical reasoning also learn through the internalization of social norms and practices, so these internalization processes are common across levels of content and sophistication (Cobb & Yackel, 1996).

Though there are obvious differences in the social contexts of mathematics students and mathematics researchers (discussed in more detail below), the two communities affect individual member participation in similar ways. Both the mathematics classroom and research community are examples of communities of practice (Wenger, 1998). Both establish expectations of individual members, determine the importance of particular problems or subjects,

and place restrictions on validity that are not determined by the individual doing the work. Authority is more distributed in the research community than it is in the classroom, but both groups (students and researchers) are participating in a larger mathematical community of practice with particular standards for operating. Social mathematical practices are necessarily shared by all members of that larger community, including both students and researchers, indicating at least some similarities in social practice between these two groups.

In addition, many of the practical actions that are useful to mathematicians are also beneficial to the learning of mathematics. Polya's (1957) four-step problem solving heuristic includes strategies for tackling each step. These include such tried-and-true general habits as "draw a picture", "introduce suitable notation", and "look for a related problem". While written with the student in mind, it is striking the degree to which Polya's strategies echo the processes of research mathematics described above. He describes generalization, representation, translating, experimenting, and the use of language and systems – all habits of mind involved in mathematics research as described by Cuoco, Goldenberg, and Mark (1996). In fact, the practical actions for mathematical work that they describe are useful for both students and mathematicians. Though the content is different, many of the same actions and reasoning processes remain consistent.

Thus, there are a variety of parallels between the processes of mathematics research and student learning. It is beneficial to think of both as jointly individual and social in nature, and many of the processes of mathematical



thinking are present (albeit at different levels of complexity) in both. In addition, the same habits of mind prove useful in both learning and researching.

### **Differences Between Mathematics Research and Student Learning**

Despite the similarities described above, the processes of mathematics research and student learning in mathematics are far from identical. For instance, there is a massive difference in the scale on which they are undertaken, since a relatively small number of people are engaged in research while millions of children are learning mathematics. While learners may be creating new mathematics for themselves, they are not creating mathematics that is new to the community at large, so the significance of the “discoveries” differs a great deal. Whereas the discoveries of the researcher matter at least to the other researchers in his or her field (and possibly have much wider implications), the knowledge created by a learner impacts only that individual, or at the most, those in his or her immediate learning context.

On the cognitive level, though the habits of mind, ways of thinking, and cognitive processes employed by mathematicians may be desirable or useful in students learning in mathematics, they may not necessarily be present. Students’ primary work is geared toward developing of these cognitive tools, while a mathematics researcher’s primary work necessitates the use of them. Moreover, the responsibility taken on by the actors in each varies widely. In the mathematics research community, authority rests in the proof argument, and a researcher or creator of mathematics assumes, with the rest of the community, the responsibilities of creation, verification, conjecturing, setting or maintaining

norms, and establishing or maintaining rules of procedure. On the other hand, in the classroom, to the extent they are not inherited from the larger mathematical culture, these are typically determined by the teacher or text (Ernest, 1998b). Even then, the determination of these mathematical norms is only one (arguably small) part of a teacher's job. A mathematician does not have to worry about teaching citizenship, encouraging lifelong learning, managing behavior, and teaching other subjects. Thus, mathematical authority is fundamentally different in the two contexts (Watson, 2008), and though the actors in each process may be required to behave in some similar ways, they hold vastly different stakes in the enterprise.

So, student learning in mathematics might be considered a simplified version of mathematics research. It contains some of the same character, behaviors, and actions, but their full power and implications are not brought to bear on school mathematics. It is, in some sense, an imitation of the larger field. The implications of the similarities and differences between the two domains will be discussed next.

### **Implications**

Though mathematics research and student learning in mathematics are not identical processes, there is enough similarity between the two to suggest that experience with one might impact a teacher's conceptions of the other. In particular, it raises the question of whether experience with mathematics research might impact teachers' conceptions of how students learn, and therefore their classroom practice. The hypothesis that it does seems to be the

underpinning of many mathematics immersion professional development programs, and the question formed the foundation for this study. In order to discuss this, I will next establish the nature of and relationships between teacher beliefs and conceptions, with particular emphasis on teachers' beliefs about student learning and how those beliefs impact practice.

## **Teacher Beliefs and Conceptions of Student Learning**

### **Introduction**

The term *belief* has a variety of meanings in common usage, and despite being a construct of great interest to researchers, there is not a generally agreed-upon research definition (Philipp, 2007). The term is closely associated with notions of knowledge, conceptions, and values, among others. Since the definitions of and relationships among all of these terms are ill-defined and unclear, it is necessary to discuss more precisely how they were used in the design, implementation, and interpretation of this particular study. The following sections are devoted to that purpose.

### **The Meaning of Conception**

Some researchers have used the term *conception* in reference to cognition or cognitive schemes, while others use it in a more affective sense (Andrews & Hatch, 2000), frequently depending on the research design and goals. Still others have chosen to distinguish between beliefs and conceptions as fundamentally different cognitive constructs (Furinghetti & Pehkonen, 2002). Thompson (1992), in a review of research on beliefs, employed the terms *conception* and *belief* somewhat interchangeably, but generally used *conception*

to mean a “more general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like” (p. 130). I will use the term *conception* in a very broad sense consistent with Thompson (1992) and Lloyd and Wilson (1998), who used it “to refer to a person’s general mental structures that encompass knowledge, beliefs, understandings, preferences, and views” (p.249).

For the purposes of designing, implementing, analyzing, and discussing this study, an individual’s conception of a particular subject includes all of their beliefs, values, and knowledge<sup>13</sup> about that subject. Thus, *teacher conceptions of student learning* include all of those beliefs, values and knowledge that concern student learning. These may include, but are not limited to: subject-matter and pedagogical content knowledge, knowledge and feelings about students, knowledge and beliefs about effective mathematics teaching, excitement or disillusionment about work with students, values that assign importance to learning goals, hypotheses about students’ current knowledge, and theories on mathematics learning. In addition, these conceptions are sure to overlap with conceptions about mathematics as a discipline and about their role as a teacher, among others. Using the term in this broad, inclusive sense provides a vehicle for discussing all the more specific constructs (beliefs, knowledge, values, etc) and how they interact together as a whole. In order to

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<sup>13</sup> I choose these three terms because of their prevalence in the literature and the way in which they encompass all the elements of a conception as described by Thompson (1992) and Lloyd and Wilson (1998), who are quoted above. All three will be properly defined and the relationships between them described below.

do so, I clarify these terms and the relationships among them in the following sections.

### **Beliefs and Knowledge**

There is general agreement on defining *beliefs* as “psychologically held understandings, premises, or propositions about the world that are felt to be true” (Richardson, 1996, p. 103). In common usage, “I believe . . .” claims are typically weaker than claims about knowledge because claims of knowledge convey an assumption that the claim has been or can be verified (Wilson & Cooney, 2002). One could, then, distinguish between knowledge and belief by requiring knowledge to carry some truth condition – as justified, or justifiably, true belief. However, in rejecting a Platonist philosophy of mathematics, that which is considered “truth” becomes fallible, making a division between beliefs and knowledge based on verification extremely problematic, particularly so in mathematics. What is seen as indisputable may change over time and depend on the current system (Lakatos, 1976; Thompson, 1992).

Reflecting the shifting nature of truth requirements, Thompson (1992) changed the requirement for distinguishing between beliefs and knowledge from verification to existence of a general agreement on how verification might be achieved. With this definition, what is now considered true knowledge may not be categorized as such at some other time. Even more recently, definitions of beliefs and knowledge have taken on an increasingly subjective tone. Philipp (2007) offered this distinction:

As a researcher, I have found the following stance useful when I attempt to understand how a person holds a particular conception:

A conception is a *belief* for an individual if he or she could respect a position that is in disagreement with the conception as reasonable and intelligent, and it is *knowledge* for that individual if he or she could not respect a disagreeing position with the conception as reasonable or intelligent. By this definition, agreement upon what constitutes 'a reasonable, intelligent position' is unnecessary (p. 267, italics in original).

In Philipp's formulation, two individuals could hold the same conception, but one could hold it as a belief and the other as knowledge. Indeed, two people with opposing beliefs on a particular subject may find more common ground than two holding similar conceptions, but one holds it as knowledge and one holds it as a belief. This definition acknowledges that *how* an individual holds a particular conception is as important as what conception is held (ibid, 2007). Thus, in considering teacher conceptions of student learning, I was able understand and describe what the conception was and how important, how fundamental, that conception is to shaping the ways in which teachers interact with students. It also allowed me to acknowledge and describe any individuals who held knowledge that is unconventional or discordant with general agreement.

### **Beliefs and Values**

Though the terms *belief* and *value* are often used interchangeably (Bishop, Seah, & Chin, 2003), some researchers have considered values to be a specific type of belief. For example, Philipp (2007) defined values to be core, guiding beliefs, that is, "belief *in*" something (as opposed to "belief *that*" statements characteristic of secondary beliefs). Similarly, Rokeach (1968, cited in Bishop, et al., 2003) viewed values as enduring beliefs, and Raths, Harmin, and Simon (1987) identified several attributes that a belief must satisfy in order to

qualify as a value. They characterized values “as beliefs that one chooses freely from among alternatives after reflection and that one cherishes, affirms, and acts upon” (Philipp, 2007, p. 266).

Though some researchers have chosen to distinguish between beliefs and values as different constructs<sup>14</sup>, I choose to blend Philipp’s (2007) and Raths, et al. (1987) notions of the relationship between beliefs and values by considering values as strongly held, core beliefs satisfying the first three requirements described by Raths, et al. (they are chosen freely from among alternatives after reflection, cherished, and affirmed), with one exception. The final requirement, that a value is acted upon, is relaxed because situations often arise where competing values are present. In such a case, not all of them may drive action. Taking this stance clarifies the language and provides a finer categorization of teacher conceptions. By distinguishing between beliefs *in* (values) and beliefs *that* (other beliefs), an increasingly nuanced view of teacher conceptions is permitted to emerge.

### **A Note on Systems**

A *belief system* has been a popular term used to refer to the various ways the interrelationships among beliefs are structured (Philipp, 2007; Thompson, 1992). The term allows researchers to classify beliefs as they relate to one

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<sup>14</sup> For example, in their review of literature on values in mathematics education, Bishop, et al. (2003) noted that beliefs are typically true/false judgments about particular statements, objects, or subjects, whereas values lie on a desirable/undesirable continuum and are therefore less context-dependent than are beliefs.

another and to discuss their arrangement in an overall scheme. Beliefs are related to each other according to the quasi-logicism of the individual holding the beliefs, the degree of conviction with which they are held, and the ways in which they cluster together (Furinghetti & Pehkonen, 2002; Green, 1971; Thompson, 1992) and the notion of a belief system has allowed researchers to describe these relationships. One way of doing so has been to label beliefs as *primary* and *derivative*, where derivative beliefs are held as a consequence of holding primary ones. In addition, the degree of conviction with which a belief is held has been highlighted by viewing beliefs as *central* (most strongly held) and *peripheral* (less strongly held). Note that I distinguished between beliefs and knowledge based on how a conception is held by an individual, so the distinction between central and peripheral beliefs based on similar criteria allows me to consider the degree to which those conceptions that are held as beliefs are truly negotiable for an individual in greater detail. Furthermore, the definition of value I adopted earlier characterizes values as central beliefs, but the terms *primary*, *derivative*, and *peripheral* are useful belief labelings for the work at hand. Thinking about beliefs as clustered acknowledges that beliefs exist in interrelated groups that may or may not be influenced by other groups (Furinghetti & Pehkonene, 2002; Green, 1971; Thompson, 1992).

In many ways, this notion of a *belief system* is similar to the notion of *conception* I described earlier. Conceptions, however, encompass systems of beliefs, knowledge, and values surrounding a particular subject, whereas belief systems may transfer across subjects. Thus, I distinguish between the two as



different but related constructs and will use *belief system* to refer exclusively to beliefs on their own, independent of relationships to knowledge, whereas the term *conception* will be used to discuss the two together. Equipped with these distinctions between beliefs (as part of belief systems), knowledge, and values, all of which are encompassed by conceptions, the literature on teacher conceptions of student learning in mathematics will be explored. This will set the stage and provide a research context for the investigation of the research questions.

### **Research on Teacher Conceptions of Student Learning**

#### **Research on the Importance of Teachers' Conceptions of Student Learning**

Recent mathematics education research efforts have highlighted the variety of factors at play in effective teaching. Models of teacher knowledge have been expanded to include not only knowledge of subject matter and knowledge of classroom practice, but also knowledge of students, curriculum, and content that is specific to the work of teaching (Ball, Hill, & Bass, 2005; Shulman, 1986). In particular, a connection between the beliefs and knowledge of teachers and their classroom practice has been established by a variety of researchers (cf., Ernest, 1989; Escudero & Sánchez, 2007; Philipp, 2007; Thompson, 1992). Beliefs and knowledge have been shown to impact the use and implementation of curriculum, instructional goals, use of technology (Philipp, 2007), and moment-by-moment decision-making (Aguirre & Speer, 2000), among others.

Thus, research indicates that teacher conceptions play a key role in the complex interaction of factors that ultimately determine what happens in the

classroom. For this study, I focused on teachers' conceptions of student learning. The choice of this category grows from the previously discussed parallels between mathematics research (the experience undertaken by participants in the study) and student learning of mathematics (the professional work of the participants).

The view of learning as a jointly social and individual task suggests that teachers should attend to the ways in which students learn in order to best encourage knowledge construction, and there is a general agreement on the value of basing instructional decisions on students' thinking (Lloyd, 2002; Warfield, 2001). Schifter (2001) pointed out that identifying the mathematical understandings of students is a necessary skill for effective teaching, but that teachers' attentions to those understandings vary. In fact, some research has indicated that teachers do not use knowledge of students when making instructional decisions (Carpenter, Fennema, Peterson, & Carey, 1988; Clark & Peterson, 1986), or base their practice more on beliefs about mathematics as a discipline than on their beliefs about pedagogy (Raymond, 1997). However, programs that focus teachers' attention explicitly on the mathematical understandings of their students, such as SummerMath for Teachers (Schifter & Fosnot, 1993; Schifter & Simon, 1992) and Cognitively Guided Instruction (Carpenter, et al., 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Carpenter, Fennema, & Franke, 1996; Fennema, Franke, Carpenter, & Carey, 1993; Fennema, et al., 1996) have demonstrated that knowledge of children's thinking impacts the beliefs and practice of mathematics teachers. Furthermore,

elementary teachers that have a cognitively-based perspective on learning tended to structure lessons differently, using more word problems to introduce concepts and teaching fewer number facts, and their students demonstrated increased word problem solving capabilities (Peterson, Fennema, Carpenter, & Loef, 1989). All of these studies indicate a relationship between teachers' knowledge of children's mathematical thinking and student achievement, but stop short of identifying explicit relationships between changes in beliefs and changes in instruction, stating that these relationships are too complex and idiosyncratic to be understood in a comprehensive way (Fennema, et al., 1996). By focusing on case studies of individuals, other studies have managed to identify some such relationships.

Beswick (2007) reported a case study of two secondary teachers in Australia whose classrooms were "consistent with constructivist principles" (p. 103). One of these teachers, Jim, expressed a view of learning that was student-centered and could be influenced but not controlled by a teacher. The beliefs that "students' learning is unpredictable" and "all students can learn mathematics" (p. 108) underpinned his practice, both as general guiding principles for planning and as he made moment-to-moment decisions. The other teacher in the study, Andrew, believed "the teacher has a responsibility actively to facilitate and guide students' construction of mathematical knowledge" (p. 113). Andrew's system of beliefs guided his practice in the same way Jim's beliefs guided his, and both presided over classrooms that were consistent with constructivist principles.

Escudero and Sánchez (2007) conducted a similar study of two teachers in Spain as they taught a specific lesson on Thales' theorem<sup>15</sup> and similarity. The first teacher, Ismael, approached the lesson guided by the beliefs that “students come to see situations that would allow them to reveal mathematical meanings” and that learning was an active process. These beliefs, interacting with his subject-matter knowledge and conceptions of mathematics, prompted him to provide students with problem situations to instigate learning and instigated a willingness to diverge from his established lesson plan. The second teacher, Juan, on the other hand, saw teaching as providing mathematical ideas to students. He used sequential steps of providing definitions and theorems followed by monitored practice verifying these results and did not vary from his pre-formed lesson plan. These actions were driven by the belief that students “believe things more easily after having verified them” (p. 324), and that learning proceeds in a linear fashion, each new piece of knowledge building on previous ones in a logical and organized manner.

Tzur, Simon, Heinz, and Kinzel (2001) wrote an account of Nevil, a fifth grade teacher in the United States, as he taught four lessons on the long-division algorithm. He valued student participation and collaboration, claimed to stress connections between basic ideas of arithmetic and the algorithms they were learning, and differentiated between understandings of individual students. The

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<sup>15</sup> Stated in the paper as: “If several parallel straight lines are cut by two transversal lines, the ratio of any two segments of one of these transversals is equal to the ratio of the corresponding segments of the other transversal” (Escudero & Sánchez, 2007, p.314-315).

belief that students construct their own knowledge through participation in classroom activities guided his approach to class structure and lesson development in a general sense, and his specific conceptions of how students learn specific concepts guided his particular actions in class. For instance, he believed that once students solved division problems using a partitive method<sup>16</sup> (with base ten blocks), the connection to the algorithm would become obvious. This conception of how students learn was the key factor in his structuring of the lesson for the class and his interactions with individual students. He was unwilling to believe that students saw absolutely no connections between the two. Tzur, et al., postulated that Nevil's own mathematical understanding and experience shaped his conceptions about how students will learn, perhaps more than he realized.

Cavey, Whitenack, and Lovin (2006) conducted a microanalysis of an Algebra I teacher's (Mrs. Lowe) teaching of slope. They considered the possible trajectories of response to a student's question, highlighting the mathematical ideas on which Mrs. Lowe could have potentially drawn. The authors use this as an opportunity to coordinate two theoretical perspectives in order to develop a framework for retrospective analysis of the teacher's classroom work. The case study also highlighted how the construction of possible trajectories for student learning guides teacher practice, and that such constructions are necessarily guided by the teacher's conceptions of how students learn.

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<sup>16</sup> That is, interpreting  $M \div N$  as the number of units in each group if  $N$  groups are formed from the total  $M$ .

An emphasis on possible trajectories of student learning is also seen in Simon's (1995) *Mathematics Teaching Cycle* framework. Based on analyses of teaching episodes, he developed a framework for conceptualizing mathematics teaching from a social constructivist perspective. It cycles through a repeating series of phases. The teacher's *assessment of students' knowledge* is filtered through the *teacher's knowledge*, which consists of several domains. Simon specifically mentions the teacher's "knowledge of mathematics", "knowledge of mathematical activities and representations", "hypothesis of students' knowledge", "theories of mathematics learning and teaching", and "knowledge of student learning of particular content" (p. 137). All of these domains of teacher knowledge<sup>17</sup> interact to create a *hypothetical learning trajectory* (HLT) on which the cycle hinges. The HLT is the anticipated path for student learning, "a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136). It consists of three parts, the teacher's learning goal, plan for learning activities, and hypothesis of learning process (ibid, 1995; Simon & Tzur, 2004). In this model, the teacher plans for a lesson by generating a HLT, and it is adjusted as the lesson goes on or from lesson to lesson as the teacher's assessment of students' knowledge is filtered through his or her knowledge and conceptions.

In a similar vein, Realistic Mathematics Education (RME) is an approach to instruction design developed principally in the Netherlands. A teacher who

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<sup>17</sup> Note that many of these domains fall inside what I have termed *teacher conceptions of student learning*.

develops mathematical tasks using RME is guided by several heuristics, two of which I highlight here because of they concern the ways teacher conceptions of student learning impact classroom practice. First, *didactical phenomenology* considers the learning trajectory of students as they might understand mathematical ideas and reinvent them for themselves. *Emergent models* anticipate how student understandings might develop, become more sophisticated, and begin to fit into conventional forms of mathematical reasoning (Gravemeijer, 1994). RME encourages basing all of this in the historical development of mathematical ideas, but, in practice, teachers are often unaware of this historical background and instead base it in their own understandings, conceptions, and personal concept developments (Cavey, et al., 2006).

The case studies described above, along with Simon's (1995) Mathematics Teaching Cycle and RME highlight the important role that teacher conceptions of student learning play in lesson development. They emphasize that lesson development is necessarily filtered through and informed by the individual teacher's conceptions of student learning. In a study of student teachers in Israel, Penso and Shoham (2003) investigated the arguments they used to justify pedagogical decisions. The researchers found that these teachers referenced the thinking and needs of learners far more when discussing decisions made during planning than they did when discussing decisions made during performance, while references to factors such as content and environment remained relatively unchanged. Based on that data and earlier research (Carpenter, Fennema,, Peterson, & Carey, 1988; Clark & Peterson, 1986), it

appears that teachers' conceptions of student learning play a small role in moment-to-moment decision making but a larger role in planning and preparation for teaching, the work that determines the initial direction of and sets the stage for in-class practice. This is true even when teachers' conceptions and expectations don't match up with the reality of student understandings (Nathan & Koedinger, 2000a, 2000b).

In this study, I made use of Simon's (1995) hypothetical learning trajectory framework to gain access to the conceptions teachers hold as they formulate the sequence and design of mathematical tasks and lessons for their students. I also drew on the important relationship between teacher conceptions and their structuring of practice established by the research reviewed above.

### **Relationships Between a Teacher's Own Learning and Beliefs About Student Learning**

Once a learner constructs a scheme of mathematical understanding, it becomes difficult to conceive of how one might approach these ideas without that particular scheme. Cobb (1989) described this as follows:

Once we have made a mathematical construction and have used it unproblematically, we are convinced that we have got it right – it is difficult to imagine how it could be any other way. (p. 33)

For the teacher in particular, breaking that knowledge back into parts and seeing how much detail is involved in the construction of it is a difficult proposition (Dreyfus, 1991; Thurston, 1990). This has been referred to as an *expert blind spot* (Nathan & Koedinger, 2000a; Nathan and Petrosino, 2003), where educators with significant conceptual understanding use their own understanding



as the guiding principle for instruction rather than focusing on how students might develop those understandings. Furthermore, teachers can impute their own mathematical understandings to their students. For instance, fifth grade teacher Nevil, mentioned above, saw student learning of division through his the lens of his own, and did not seem to realize the impact his conceptions were having on his expectations for his students (Tzur, et al., 2001).

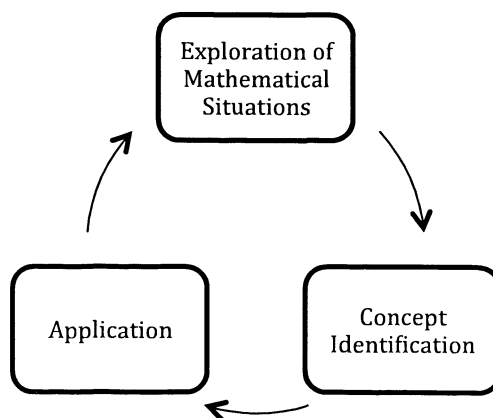
Primary beliefs – those most resistant to change – often develop during a teacher's time as a student (Clark and Person, 1986; Thompson, 1992). Indeed, two primary sources of beliefs are emotion-packed experiences and cultural transmission (Ambrose, 2004), both of which are characteristic of an individual's school experience. Gates (2006) sees beliefs as part of a sociological construct rather than simply a cognitive one, and social experience, particularly the classroom, heavily influences them. Thus, the classroom approach that teachers take is often influenced by their own experiences as learners. Raymond (1997), for instance, described Joanna, a beginning teacher, who believed that mathematics was a fixed, static collection of unrelated facts. This belief, formulated during her experience as a mathematics student, influenced her teaching style more than any other factor. Despite graduating from a reform-oriented teacher education program, Joanna's practice reverted to her own experience as a learner – the beliefs she developed there were primary and extremely resistant to change.

When teachers come to an understanding through a particular activity, they often attribute it as a property of that activity and thus try to recreate it for

their students (Heinz, Kinzel, Simon, & Tzur, 2000). In other words, a teacher is likely to see the activities that help guide him or her to understanding as valuable for all learners of the content, and will often explain or predict student difficulties using their own experiences (Badertscher, 2007). Their view of how students will construct a concept is contingent upon, or at least largely shaped by, how they themselves have experienced the construction of the concept. When teachers reflect on learning experiences and seek out connections to other knowledge and to their work in the classroom, shifts in beliefs are more likely to occur (Cooney, Shealy, & Arvola, 1998).

Simon (1994), based on work with both in-service and pre-service teachers, formulated a series of Learning Cycles for mathematics teachers that together form an overall *Teacher Learning Cycle*. The first cycle is for *learning mathematics*, and consists of exploration of mathematical situations, which prompts teachers to engage in concept identification, followed by application of those concepts to new situations or existing conceptions, which then leads to further exploration, as shown in Figure 1.

FIGURE 1: *Simon's (1994) Teacher Learning Cycle*



Each of the 5 other interrelated cycles has this cycle embedded within it. These learning cycles are for *developing knowledge of mathematics, developing theories of mathematics learning, understanding students' learning, instructional planning, and teaching*. These cycles are similar to the one shown in Figure 1, but “exploration of mathematical situations” is replaced by the *Teacher Learning Cycle*. That is, the cycle through which teachers develop each of those categories of knowledge begins with mathematics learning, moves to concept identification in that particular area, and then application which leads back to the first learning cycle. Each learning cycle also influences all the others, but the presence of personal mathematics learning as a key component in the development of all aspects of knowledge for teaching highlights the important role that a teacher’s personal experience as a learner plays in shaping his or her conceptions about student learning.

### **Categories of Teacher Beliefs**

Having established that teacher conceptions of student learning are important for the planning and sometimes the implementation of practice, and having noted the important role that a teacher’s own experience as a learner plays in shaping those beliefs, I now turn to discuss the various ways researchers have categorized those beliefs about student learning. These can provide a framework for discussing what teachers actually believe. Ernest (1989) proposed organizing teachers’ models of learning mathematics around two key constructs:

A view of learning as the active construction of knowledge as a meaningful connected whole, versus a view of learning

mathematics as the passive receptions of knowledge; The development of autonomy and the child's own interests in mathematics versus a view of the learner as submissive and compliant. (p. 23)

Thus, there are two axes structuring teacher models for learning; one concerned with knowledge formation in learners, and the other concerned with the autonomy of the learner. Though Ernest did not claim to characterize teachers' models entirely, he proposed that these axes are the key aspects and described six simplified models of learning mathematics, as follows:

- [1] - child's exploration and autonomous pursuit of own interests model
- [2] - child's constructed understanding and interest driven model
- [3] - child's constructed understanding driven model
- [4] - child's mastery of skills model
- [5] - child's linear progress through curricular scheme model
- [6] - child's compliant behavior model (p. 23, numbers added)

The first represents an extreme position on both axes - that knowledge is actively constructed and children can do so autonomously through exploration of their own interests. The second still holds that knowledge is actively constructed, but lessens the emphasis on autonomy, allowing that knowledge is driven, but not entirely directed by, individual interests. The third model is motivated entirely by a belief in individually constructed knowledge with no attention paid to the autonomy of the child as constructor. Thus, the teacher could introduce the activities and topics without considering the child's interest. The fourth model holds that learning mathematics consists of mastering skills, while the fifth is driven by the belief that mathematics learning proceeds linearly. Both treat the learner as more passive than in previous models, but the fourth model does

stress individual mastery. The sixth model represents the opposite extreme from the first model, that children are passive, compliant receivers of knowledge.

Another approach has been to categorize teacher perspectives on their work in general, encapsulating conceptions of both mathematics and mathematics learning. Simon, Tzur, and colleagues (Simon, Tzur, Heinz, Kinzel, & Smith, 2000; Tzur, et al. 2001) formulate three such perspectives, *traditional*, *conception-based*, and *perception-based*. A traditional perspective on teaching and learning is based on the idea that learning is accessing an external reality and that knowledge is transmitted to students. A conception-based perspective, on the other hand, holds humans have no access to any reality outside of that which they experience, and learning is the “building up and continual transformation of one’s conceptions” (Tzur, et al., 2001, p. 247). The researchers claim that though this perspective has become popular and influential among researchers and teacher educators, and has influenced recent education reform efforts, it is not held by very many teachers (Simon, et al., 2000). Rather, many teachers hold a perception-based perspective, which is a middle ground between the previous two. It couples the Platonist view of mathematics found in the traditional perspective with an emphasis that students come to understanding mathematics through their own experience with it.

Kuhs and Ball (1986) similarly identify “four dominant and distinctive views of how mathematics should be taught:”

1. *Learner-focused*: mathematics teaching that focuses on the learner’s personal construction of mathematical knowledge;

2. *Content-focused with an emphasis on conceptual understandings*: mathematics teaching that is driven by the content itself but emphasizes conceptual understanding;
3. *Content-focused with an emphasis on performance*: mathematics teaching that emphasizes student performance and mastery of mathematical rules and procedures; and
4. *Classroom-focused*: mathematics teaching based on knowledge about effective classrooms. (p. 2)

A *learner-focused* view of mathematics teaching emphasizes students' active construction of knowledge and the authors closely associate it with a constructivist view of learning. Teachers with this view give students responsibility for their own ideas and use assessment to determine how well personal ways of understanding mesh with the commonly-shared mathematical meaning of a concept. Teachers with a *content-focused with an emphasis on conceptual understandings* view let mathematics content, rather than student understandings, determine and organize the subject matter and classroom activity. The authors posit that this is driven by a belief that the body of mathematical knowledge is fixed and uniform. Subject matter is organized similarly for teachers with a *content-focused with an emphasis on performance* view, but the focus is on procedural fluency for students. Mathematics is seen as governed by and built on rules, and proficiency is demonstrated by automatized procedures. The final view of teaching identified by these authors, *classroom-focused*, does not address the content or espouse a particular theory on learning. Rather, this view centers on the assumption that students will learn best when classrooms are structured according to principles of effective instruction established by process-product studies, tradition, and common sense (ibid., 1986).

Other researchers have categorized beliefs about student learning of particular topics (cf., Nathan & Koedinger, 2000a, 2000b), but, as shown in the frameworks described above, most attempts to categorize teacher beliefs about student learning as a whole categorize these beliefs along a scale measuring the degree to which teachers believe students construct their own knowledge or receive it.

### **The Impact of Mathematics Immersion Experiences**

#### **Results From Previous Research Studies**

Cuoco (2001) asserted that a research-like experience in mathematics (an experience that is perhaps not original work in mathematics, but is original to the learner and simulates the mathematics research process) would have a developmentally powerful influence on teachers. Organizers of mathematics immersion programs have undertaken the task in a variety of ways, with varying research and outcome goals, and the programs have demonstrated range of impacts on the participants. Since this project investigated the impact one such program had on its participants, it is appropriate to consider what impacts similar programs have exhibited. In this section, I will do just that.

Teacher educators at Illinois State University instituted the Teacher-Scholar Program, wherein pre-service secondary teachers participated in authentic mathematics research guided by a working mathematician and instigated by a course. These students did not just simulate the mathematics research process, they actually engaged in original research in graph theory, number theory, combinatorial design theory, and mathematical and statistical

modeling. Several students obtained original results from their work. The organizers focused their research on how the experience impacted the participants' beliefs about mathematics as a discipline. Using Likert-scale surveys, they found that some aspects beliefs did indeed change over the course of the semester's participation. In particular, participants moved from the belief that mathematics is a rigid, previously determined collection of facts toward the view that mathematics is a problem-solving discipline grounded in the work of individuals. The researchers also gathered evidence that some aspects of beliefs about mathematics learning changed, moving from valuing in learning from computation toward valuing student problem-solving (McCrone, et al., 2008).

Researchers at the University of Maryland created a course for graduate students in mathematics education that offered the students the opportunity to participate in mathematics work and drive content development more than is possible in a typical graduate mathematics course. A research-like project was one component of this class. Students began their projects with a personally-motivated open (to themselves, not to the field in general) question or problem grounded in secondary-level mathematics. With mathematicians as resources, the students explored their problems in much the same way a mathematician might; conjecturing, guessing, creating, and proving (Chazan, et al., 2007). The results reported by students in the course were mostly affective: despite an initial discomfort born of reluctance at approaching mathematics in an unfamiliar way, many students reported experiencing a sense of excitement and freedom at



being in control of their own learning. The ability and opportunity to independently explore mathematics was a liberating experience for many, and it shifted the ways they assigned priority to aspects of mathematical work (ibid, 2007). Some graduate students who were interviewed about their experiences in mathematics courses (of all types) described this particular course (and its follow-up, companion course) as “influential”, “transformational”, and “inspiring” (Marshall, 2008).

Badertscher (2007) facilitated a graduate course in mathematics for middle school teachers using many of the same classroom features as the University of Maryland course, including open exploration and investigation of personally-motivated mathematical questions. She investigated how individuals’ identities and personal ways of knowing shaped and mediated those individuals’ experiences in the course. The two individual case studies highlighted present quite contrasting experiences. For one teacher, the experiences of doing mathematics differently than she had come to know it shifted her perspectives into a “more coherent whole” (p. 217). This teacher was uncomfortable with mathematics prior to the experience and described herself as disliking the subject. For her, the new way of approaching it was a renewing alternative that reorganized her thinking about the subject. The second teacher, on the other hand, was comfortable with her perception of mathematics. Mathematics, as she saw it, made sense to her, and the new experience upset the balance she had found and challenged her identity as a teacher. The experience was impactful

for both teachers, though the degree to which that impact was positive differed from the perspective of each one.

The Math in the Middle Institute at the University of Nebraska immerses teachers in mathematical work in an effort to help them develop mathematical habits of mind (see Cuoco, Goldenberg, & Mark, 1996). Case studies of three middle school teacher participants in this program indicated that, other than school situation, the professional development experience was the most important aspect of the teachers' context in determining their practice, but that *context* was a very complex and subjective construct (Smith, 2008). An action research project produced by participants in this program showed them engaged in evaluation of their own practice as a result of their participation. Another mathematics immersion experience for teachers, the Park City Mathematics Institute (PCMI), conducted interviews with teachers that indicated many developed new habits of mind through the program, began to think about optimal conditions for student learning, and reconsidered how students develop their knowledge (Stevens, et al., 2008).

PROMYS (Program in Mathematics for Young Scientists) for Teachers, an outgrowth of the original PROMYS for high school students, offers secondary mathematics teachers opportunities to explore mathematics through open-ended problem solving, principally in the area of number theory. The first of two summers exclusively focuses on this, while the second summer includes a long-term investigation of a mathematical problem guided by a mathematician, simulating the mathematics research experience. Many of the participants in this

program self-reported changes in their beliefs about the nature about mathematics and the ways they approached teaching. Changes included increased confidence in approaching both familiar and unfamiliar mathematics, and new expectations for students. The new expectations included increased confidence in student capabilities and higher expectations for what students can accomplish on their own (Stevens, et al., 2001).

These programs all value providing teachers with mathematics immersion experiences and share a common outcome of demonstrating that participants are impacted in some way. The nature of the demonstrated outcomes has been guided by the research goals of each particular project, but this model for professional development has a well-established precedent of impacting teacher conceptions.

### **Conclusion**

Up to this point, I have been building a context for the study. In doing so, I have established a social constructivist theoretical framework for student learning and mathematics; discussed the nature of mathematics research in order to identify the key aspects and characteristics of mathematics immersion experiences; defined constructs of conception, belief, knowledge, and values in order to discuss teacher conceptions of student learning; and reviewed research relevant to this work in order to establish a research background for it. With these definitions and frameworks in hand, I now turn to discussing the shape of the study – to showing how I will draw on the conceptual framework and relevant literature in order to investigate the research questions at hand.

## CHAPTER III

### METHODOLOGY

#### Introduction and Research Setting

##### Introduction

The main research question of this project was stated as follows: **How does a mathematics research experience impact teachers' conceptions of student learning?** In order to investigate this question, I used a combination of Likert-scale belief surveys and task-based clinical interviews administered to participants in a summer professional development program grounded in mathematics immersion. Observations of the summer program and the participants' classrooms complemented and informed the other data collection methods. All of these data sources were combined to develop case studies for each of the five interview subjects. Bryman (2007) cautioned that one has to be careful mixing quantitative and qualitative methods because they can rest on contradictory theoretical foundations. However, if a careful theoretical framework is used, utilizing techniques that are mutually illuminating can offer valuable insights into the problem at hand (ibid., 2007). In this case, the survey data placed the case studies in a larger context and the case studies provided a level of detail that surveys could not provide. The role of each of these methods and how they were used together is described below.

## **Research Setting**

I investigated a group of high school teachers participating in a professional development program based on mathematical immersion that has a research-like project as a critical component. For the purpose of this work, I will refer to it as the RLE program (for “research-like experience”). It was offered by a large private university in the northeast and comprised two six-week summer sessions (over the course of two years) in addition to five one-day seminars during the school year. I collected data during the summer of 2009. All participants attended a daily morning lecture on number theory, a subject area chosen because of its deep mathematical ideas and low threshold, requiring only a solid background in algebra. In the afternoon, participants worked in groups on problem sets designed to encourage “thinking like a mathematician”. The organizers made clear to the participants that they were not expected to finish the entire problem sets, which were intentionally designed to be broad in scope in order to provide multiple entry points. First-year teachers were especially encouraged to work on “numericals”, computational problems which form a basis for conjectures and generalizations. Second-year teachers were encouraged to attempt more open-ended and proof-based problems. For both groups, the organizers encouraged axiomatization and proving from axioms. Furthermore, during the summer I observed the program, the lecturer frequently emphasized that mathematicians draw on their mathematical experience to develop the formal structures of mathematics and that the problem sets mirror that process.

Counselors were available to assist and guide participating teachers in their work, and the teachers could receive graduate credits in mathematics for taking the course. The five day-long workshops during the academic year focused on pedagogy over content, though time was set aside for problem sessions. During the second summer, teachers retook the number theory course participated in small-group research projects mentored by research mathematicians. They did not produce original mathematical results, but the open-ended exploratory nature of the projects simulated mathematics research, as each group constructed results that were original to them.

The RLE program began as a residential summer enrichment for gifted high school students before expanding to include teachers, and this aspect of the program continued during the summer of my observation. The teachers and the high school students all sat in the morning lecture together and worked on the same problem sets, though they worked separately apart from the morning lecture. The teachers were exposed to the insights and thinking of the high school students as a result of the interactive nature of the lectures. Most teachers seemed to feel that the high school students understood more of the content, and on a deeper level, than they were themselves, and that the students demonstrated a level of understanding that was not typical of their own students.

### **Participants**

The participants were all teachers taking part in the summer RLE program described above, of which there were 50 during summer 2009. Twelve of these teachers were there for the second summer, and three were there for a third.

The third-summer teachers took a geometry course instead of the number theory course and, except for working in the same study rooms and attending large-group meetings, were isolated from the rest of the group. Of the 35 first-summer teachers, approximately 10 were pre-service teachers completing the program as part of combined bachelor's plus master's degree program.

I attended the first group meeting of the summer and solicited participants. Twenty-nine teachers (10 second-year) agreed to participate in the belief survey, and twenty-four of these (8 second-year) also returned the follow-up survey. The second-year teachers worked on the project in pairs, and three of these six pairs agreed to participate in the interviews. Of the six second-year participants, one moved at the end of the summer, making it impossible to complete the follow-up interview, one failed to complete the requirements of the program and did not complete the project or participate in the second interview, and another declined to participate in the final interview due to restrictions his school placed on classroom visitors. Thus, I conducted the complete series of interviews with three second-year teachers in total – Scott, Jennifer, and Joyce. In addition, several first-year teachers agreed to be interviewed. Of these, two stuck out as interesting cases. One, Emily, was a relatively new teacher (2 years experience) and her initial belief survey indicated a belief that students receive, rather than construct, knowledge. Every other participant in the belief survey indicated at least mild agreement with the idea that students construct their own knowledge, making Emily a unique case among the study participants. Another teacher, Deborah, had 16 years of experience, taught in a nontraditional setting, and her

initial belief survey indicated significant agreement with the conception that students construct their own knowledge. After observing a few days of the program, the degree to which the work of a mathematician was stressed during the first summer, and the degree to which the problem sets were designed to mimic that work, became apparent. The combination of the first-summer emphasis on mathematics research and their interesting backgrounds led me to include the Emily and Deborah in the group of interview subjects. Biographical information for each of the interview subjects and detailed information about the survey sample can be found in chapters IV and V. The sections below provide detailed descriptions of the motivation and procedures for designing and carrying out both the belief surveys and structured interviews used with these teachers.

### **Belief Surveys**

#### **Design and Implementation**

The belief surveys used a 5-point Likert scale<sup>18</sup>. Items were divided into three categories, each assessing a specific area of research interest described in the research subquestions. A copy of the survey, with codes describing the wording and category of each item, can be found in Appendix A. Items in category I assessed teachers' conceptions of student learning, specifically, the degree to which teachers believe students construct or, on the other end of the spectrum, receive knowledge. The ten items in this category were adapted from those used by Fennema, Carpenter, and Loef (1990, see also Peterson,

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<sup>18</sup> The scale is labeled as follows: 1 – Strongly Disagree, 2 – Disagree, 3 – Neutral, 4 – Agree, 5 – Strongly Agree.



Fennema, Carpenter, and Loef, 1989), and in some cases, are more closely related to versions employed by Vacc and Bright (1999) and Capraro (2005), both studies that extended the work of Fennema, et al. The ten items in category II assessed teachers' conceptions of the relationship between school mathematics and research mathematics. These items were original, but inspiration was drawn from literature concerning the relationship between the two domains<sup>19</sup>. Items in category III assessed teachers' expectations of student capabilities; that is, the degree to which teachers believe students are capable of arriving at conclusions or solutions without significant support from instructors. Two items in category I (items numbered 7 and 10 in the survey) also address this category of conceptions, and were therefore coded to category III as well as their original category. Eight original items solely assess category III. Thus, there are ten items in each category, and a total of 24 items in three different categories, broken down as follows:

TABLE 1: *Breakdown of Belief Survey Categories*

Category I only	8 items
Category II only	10 items
Category III only	8 items
Category I and Category III	2 items

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<sup>19</sup> All this literature is described in the Conceptual Framework and Literature Review. Articles by Watson (2008) and Ernest (1989), which focus explicitly on differences between school and research mathematics, proved especially useful.

All 28 items were reviewed by researchers with experience using belief surveys, and an initial version of the survey was piloted with a group of 15 in-service teachers. Revisions were made based on this data. Some items were reworded for clarification, and others were changed from negatively-stated to positively stated, or vice-versa, in order to improve the neutrality of the statement. In addition, four questions that had previously been included in two different categories were each rewritten as two separate, more specific, items.

Of the 28 items, half were positively stated and half are negatively stated, and this was true of the ten items in each category, as well. Agreement with a positively stated item in category I indicated belief that students construct their own knowledge. Agreement with a positively stated item in category II indicated belief that school mathematics and research mathematics are closely related, and agreement with a positively stated item in category III indicated an expectation that students can develop original solution methods and results with minimal teacher support. Agreement with a negatively stated item indicated the same as disagreement with a positively stated item— a belief that students receive knowledge (category I), that school mathematics and research mathematics are disjoint or nearly disjoint domains (category II), and that students need significant teacher guidance and intervention in order to develop new insights. The tendency to simply agree or disagree with all statements is minimized by including both positively- and negatively-stated items (Capraro, 2005), and the items were randomly ordered in order to avoid the appearance of patterns that respondents might be tempted to follow.

The survey was administered to all willing participants at the beginning and end of the six week summer session and the results analyzed for shifts in any of the three categories, or on individual items. In a general sense, the survey was designed to provide some snapshot of what, if any, significant changes occur over the course of the research experience. Because it surveyed the large group, this data source provided useful baseline information on categories of teacher conceptions that may have been impacted by the program. In doing so, it contributed a sensitizing context for the interviews, offering large-group results within which to understand the individual cases and indicating topics to be investigated in more detail during interviews.

However, belief surveys conducted in isolation are limited in a number of ways. There is no way of knowing whether teachers' reports are accurate, for instance, and it is difficult to know how individual participants interpret the language and grammar of each statement (Philipp, 2007). Furthermore, respondents are typically willing to provide opinions even if they've never before considered a particular matter (McGuire, 1969, cited in Philipp, 2007). In addition, conceptions of student learning are highly idiosyncratic and extraordinarily complex. Thus, it is impossible to form a very full picture of them across an entire group. It is vital that the conceptions of individuals be considered in a focused, in-depth manner. In order to combat these issues and to develop a more robust picture of how teacher conceptions are impacted, I also conducted interviews with five teachers and developed individual case studies describing their experience. The interviews themselves are described in the next

section. Peterson, et al. (1989) showed that taking survey data together with structured interviews provides a more comprehensive description of teacher conceptions, and that was the goal in this study.

### **Task-Based Interviews**

#### **General Structure**

As noted earlier, conceptions are too complex a construct to be fully outlined in general, but more significant insights may be obtained by focusing on the conceptions of particular individuals, and most investigations of teacher beliefs have taken a case study approach (Philipp, 2007). Using Simon's (1995) construct of *hypothetical learning trajectories* (see also Simon & Tzur, 2004) as a basis, the interviews were based on lesson-planning tasks designed to bring out the types of paths along which interview subjects believed student learning would proceed. Five teachers were interviewed three times each – once during the first week of the summer program, once during the final week, and once during the fall semester. Observations over the course of the summer experience and in the teachers' classrooms contributed to the development of the second and third interviews (see Glesne & Peshkin, 1992).

In developing a theoretical framework for clinical interviewing, Confrey (1981) described a clinical interview as follows:

A clinical interview aims to examine students' understandings of propositional knowledges, concepts, processes and reasons for believing in those concepts and processes. It can be based on a change perspective through which the interviewer attempts to ascertain what a student believes, why s/he believes, how s/he came to believe it and what predictions s/he might make as a result of those beliefs. Both the interviewee and interviewer assume active roles in the process, with the student for the most part

guiding the inquiry. At times the interviewer strives to clarify the meaning of the interviewee's statements, while at other times, s/he is more interactive, actively hypothesizing about the implications of the students' responses, posing new questions to test those hypotheses. (p. 14-15)

Substituting "teacher" in place of "student" in the above quote provides an excellent description of the clinical interviews that I conducted as part of this study. Lesson planning shows how teacher conceptions are enacted as they construct hypothetical learning trajectories. By structuring the interviews around lesson planning tasks, I was able to elicit teachers' conceptions of student learning and their reasons for holding those conceptions. I intentionally avoided asking teachers to explicitly describe their beliefs outside the context of their work, because beliefs described as important may not play an important role in their practice (Carpenter, Fennema, Peterson, & Carey, 1988; Clark & Peterson, 1986). Instead, the conceptions that shaped the ways they thought about student learning and thereby impacted lesson development and anticipated classroom actions were allowed to emerge.

Each teacher planned two lessons during the first interview and then revised them during the two subsequent interviews. I narrowed the scope of the interviews to one particular subject, algebra, in order to eliminate confounding factors. For instance, if a teacher were to plan a geometry lesson and an algebra lesson and the hypothetical learning trajectories they constructed indicated different or contradictory conceptions of student learning, it would be difficult to attribute this to the different subjects, the teacher having unclear or poorly-formed conceptions, or some combination of both. I chose algebra as the

subject area because it is a common subject for high school teachers to teach and is taught in a variety of ways, from step-by-step instruction to problem-based exploration. The first lesson planned by each teacher focused on either solving linear equations or solving systems of two linear equations in two variables, depending on the teacher's comfort and typical teaching responsibilities. Every teacher planned a second lesson on defining the mathematical term *function*. I chose these common topics in part because they are a common part of high school curricula, and in order to have one topic that is typically treated as mechanical and one that is typically treated as conceptual.

Participants developed a lesson plan for each topic during the first interview, then revisited this plan immediately after the summer course project and then again after returning to the classroom in order to discuss how they might alter or re-conceive the lesson. When I introduced the task to the participants, I asked them to describe their learning goals and planned instructional activities while verbalizing the reasons behind their choices. Only after they had finished discussing the lesson did I ask them to go back through it, explaining how they expected student learning to proceed, thus completing their construction of a hypothetical learning trajectory. I approached the tasks and the interviews as a whole in a conversational style, and asked participants to expound on their statements when they were unclear, illuminating, or lacking explanation according to established interview techniques (Seidman, 2006; Davidson, 2003). Confrey (1981) suggests that "a clinical interview be oriented toward concepts, processes, change, and justification" (p. 12) and the structure

allowed for just such an orientation. Below I will detail the specific content and form of each of the three interviews. The interview protocols can be found in Appendix B.

### **Initial Interview**

The first interview took place during the first week of the summer program and began with a series of biographical questions to determine the teacher's educational and professional background, comfort level with mathematics, experience with mathematics exploration, and motivation for participating in the summer program. For those teachers who were participating in the second summer of the program, I asked about their experience during the first summer and how it impacted their work during the interceding school year, if at all. This portion of the interview was very conversational. I asked open-ended questions and encourage teachers to elaborate on their answers rather than posing scripted follow-up questions or prompts (Davidson, 2003). This was intentional in order to develop rapport and establish an informal interview environment.

Following the introductory questions, I explained the tasks to the teacher, asking them to describe what they would do in the lesson and why they would do it, emphasizing that no level of detail is too great. Pilot interviews indicated that many teachers are uncomfortable creating a lesson from scratch, and providing a lesson outline as a starting point proved valuable, so for both lessons, a sample lesson was made available<sup>20</sup>. The lesson 1 sample was considered by only one

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of the six teachers, and that teacher looked at it, but decided not to use it as a starting point. Two teachers looked over the sample lesson for lesson 2. Either using the sample lesson as a framework or without consulting it, teachers developed a lesson plan for teaching the topic. Throughout, I prompted them to provide details on why they are making particular choices and what they expected from students, though those prompts became less necessary as the interview progressed.

After they wrote out the lesson, I asked the participant to talk through the lesson from the perspective of a student in order to make explicit their expectations of what students will be learning and how they will be learning it at each step. Finally, I asked what it would take to consider the lesson a success in order to understand how they define success for their students. The process was repeated with a lesson on the definition of *function* in order to see how consistent teachers' guiding conceptions were across topics. In all but one case, the second lesson was discussed during a second meeting that took place a day or two later. The interviews were recorded digitally and transcribed prior to the second interview. The lesson plans that were developed during the first interview were typed out for use during the second interview.

### **Second Interview**

The second interview was conducted during the final week of the summer RLE program. Analysis of the initial beliefs survey and observation over the

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<sup>20</sup> These sample lessons can be found in Appendix C.



course of the summer informed the interviews, so the precise structure was sensitive to the realities of the participants' experience and to the themes that emerged in the first interviews and over the course of the summer (Glesne & Peshkin, 1992). Introductory questions encouraged the teachers to talk about their summer experience, focusing on the mathematics they learned, what they discovered about the nature of mathematics, whether they were surprised by what they were able (or unable) to accomplish, and what, if any, influence social aspects of the program had on their work. I also asked each teacher if they thought much about their students over the course of the summer and presented a hypothetical situation wherein a student asked them about the nature of mathematics research in order to bring their conceptions about this process to light.

The second half of the interview focused on revisiting the lesson plans developed in the first interview. Those lessons, typed and organized, were a framework that teachers could use to reaffirm, alter, or redevelop their lessons. Again, I encouraged the teachers to explain the reasoning that underlies their choices and to then construct hypothetical paths for student learning. By presenting them with the lessons they had designed previously, any alterations in conceptions or expectations could be discussed directly, and, through questioning, the reasons behind any changes (or lack thereof) were explored. After discussing the hypothesized learning trajectories, I again asked the participants to define success for the lesson. As before, the interviews were

- recorded and transcribed prior to the third interview. The (possibly revised) lesson plans were typed out for use in the final interview.

### **Third Interview**

I conducted the third and final interview in late October or early November 2009 in each teacher's school. As with the second interview, analysis of the belief surveys and of the first two interviews informed and shaped these conversations. I first observed the teachers' classroom over the course of one or two lessons. Two teachers introduced me to the class, while the others did not acknowledge my presence to the students in the class. In every case, I simply observed from the rear of the classroom, taking notes and watching students interact with each other and with the teacher. The first part of the interview consisted of a discussion of the observed lesson in light of the summer experience – the reasons that teachers made particular choices during the class and whether the research experience played some role in the teachers' approach to the lesson. I also re-posed the hypothetical situation wherein a student asks for a definition of mathematics research. Secondly, we revisited the lesson plans that the teacher developed in the first two interviews, and they were again asked to make any alterations and discuss how student learning might proceed through the lessons. By following teachers back into the classroom and revisiting their previous work, I hoped to understand how resilient the summer experience proved to be upon returning to the classroom.

Finally, I asked each teacher to briefly respond to each of two statements in order to assess their conceptions of the relationships between research and school mathematics. These were restatements of items from the belief survey:

- 1) When mathematicians do mathematics, they are doing something fundamentally different than when students do mathematics
- 2) The thought processes involved in learning high school mathematics and those involved in researching mathematics are the same.

When viewed in conjunction with their responses to the corresponding survey items, these prompts elicited details about teachers' conceptions of the relationships between research mathematics and school mathematics. The interviews were again transcribed for analysis.

## **Analysis**

### **Survey Data**

The belief survey data was analyzed using standard statistical analysis techniques. Positively-stated items were scored exactly according to the response on the survey (1 for Strongly Disagree, 2 for Disagree, etc.). Since disagreement with a negatively-stated item indicated the same conception type as agreement with a positively stated item in the same category, response  $x$  to a negatively-stated items were given a score of  $6-x$ . This provided a consistent scale on which to measure teacher responses. Cronbach's alpha and standardized alpha were utilized as reliability measures in order to determine the internal consistency of each category on both the pre-test and the post-test. Alpha scores also permitted the elimination of items that lowered the reliability of

the categories. Ultimately, five items were eliminated from consideration with their category due to their adverse effect on the alpha values of their category. Paired *t*-tests using the data from the beginning and end of the summer were used to examine differences in responses for each category, and on each item. The data gleaned from the beliefs surveys provided baseline information that informed the in-depth case study and offered snapshots of the entire group's conceptions about student learning immediately before and immediately after participation in the mathematics immersion program.

### **Interview Data**

Analysis and interpretation of the interview data took place in several phases over the course of several months. The first level of interpretation occurred as the interviews were transcribed (Kvale, 1996; Seidman, 2006). Since I was analyzing the interviews for teacher conceptions rather than performing a discourse analysis, I transcribed the interviews with the goal of faithfully capturing the exact words used and meanings implied by the participants, but not necessarily every pause, "um . . .", and "ah . . ." verbatim. I attempted to construct a valid transcription from conversation to written word, capturing the words without losing the meaning and implications of those words in a possibly incomprehensible transcription or one that creates an unnecessarily negative impression of the interviewee (Lapadat & Lindsay, 1999; Poland, 2002).

After transcription, the next step in analysis was to clarify the material by categorizing and simplifying the transcribed interviews in manageable chunks. I first examined the series of three transcripts for each participant in order to

describe each individual's case. I developed an open coding scheme that developed from the transcripts themselves (Kvale, 1996) and used the constant comparison method (Glaser, 1965) to develop themes. The codes categorized background information and comments that pertain to the research subquestions described above. Specifically, codes were helpful for designating comments that indicated a belief that students construct their own knowledge and those that indicate a belief that students receive knowledge. Also, comments that convey high or low expectations for what students are capable of were of great interest, as well as any references to the interviewee's own learning when discussing student learning or to mathematics research or their work on the project. Codes also proved useful for identifying comments that discussed the nature of mathematics and mathematics research. After analyzing each participant's series of interviews, I wrote up a case study for each, then examined and compared all six case studies in order to find general themes.

### **Putting It All Together**

Case studies allow one to delve deeply into a small number of examples in order to gain maximum knowledge about those examples. Chosen carefully, cases can provide enough information about a phenomenon to allow for some generalization. This can take place through the accumulation of enough cases to allow for patterns to emerge. On the other hand, even a small number of critical or unusual cases can often offer insights that could be missed by large-scale studies (Flyvbjerg, 2004; VanWynsberghe & Khan, 2007). The goal in this study was to study enough cases to allow for cross-case comparison, acknowledging

that the numbers were not large enough to permit blanket statements about the group at large. At the same time, however, the varied backgrounds and experiences of the individuals studied and the in-depth nature of the knowledge gained permitted for conclusions to be drawn regarding the impact of the experience. Furthermore, the variety of the individuals allowed each case to illuminate the others, highlighting differences and similarities in their reaction to the RLE program and their beliefs related to it. Thus, the final step of analysis was to take the case studies together with the survey data and interpret it through the conceptual lens I applied to the work.

Kvale (1996) claimed that there is no “one way to find the meaning of interviews,” and that was certainly the case for this data. Thus, the techniques used to draw meaning from them developed as the analysis went on. I first analyzed the cases individually in order to allow the dominant themes emerge for each individual. Then, drawing upon techniques described by Miles & Huberman (1994), I compared and contrasted the themes that emerged from the different case studies and clustered themes into groups. Keeping in mind the goals of the research project, themes that related to how individuals use experience with mathematics research when considering student learning, and what role the learning provided by such an experience plays in the development of teacher conceptions were of particular interest. The conceptual framework described in Chapter II served as lenses through which to view and interpret the information that emerged.

Finally, I considered the group of individual case studies together with survey data from the larger group in order to draw some broader conclusions about the impact of research experience on teacher conceptions of student learning and how that impact transferred to the classroom. The previous research literature described in Chapter II informed the entirety of the process, and, once themes emerged from the data, I drew upon that research in order to make conclusions about the impact of this research-like experience for teachers and the generality and generalizability of the results.

The next chapter consists of the five individual case studies. Each contains relevant background information on the interview subject; a summary of their summer learning experience, with emphasis on those aspects that they identified as meaningful or important; a summary of their conceptions of student learning as revealed by the belief inventories and hypothetical learning trajectories; their impressions of mathematics research and its relationship to student learning; and a discussion of how each of these elements interacted over the course of the summer. By describing each of these, I will point out both what changes were evident in their conceptions of student learning and how those changes came about. In Chapter V, the aggregate belief survey data will be reported, and the case studies will be analyzed as a whole in order to draw conclusions about the impact of this experience on teachers' conceptions of student learning.

## CHAPTER IV

### CASE STUDIES

#### Introduction

##### The Role and Value of Case Studies

Case studies have been characterized as everything from a methodological choice to a choice of what is to be studied. In much research making use of them, as in this study, the actual role falls somewhere in between (Schram, 2006). I view the case studies not as a methodological tool – the interviews and observations were the method by which data was collected – but rather, as VanWynsberghe and Khan (2007) proposed, a heuristic. They describe their meaning of heuristic, which I adopt in this case, as follows: “An approach that focuses one’s attention during learning, construction, discovery, or problem solving” (p. 2). The case study approach focused the analysis on the experience and conceptions of individual teachers as they participated in the RLE program. Only after considering each participant as an individual was an effort made to delineate themes and understand their experiences together with the other case studies and the information from the belief survey.

Thus, each of the cases contained in this chapter describes the relevant background of each individual, summarizes their responses on the belief survey, and outlines the hypothetical learning trajectories they constructed in order to



discuss and illustrate the conceptions of student learning that played a role in their lesson development. The changes in their lessons, hypothetical learning trajectories, and discussions are noted in order to show how their conceptions changed over the course of the series of interviews. Furthermore, the individual's descriptions of their experience in the program and their conceptions regarding the relationship between mathematics research and student learning are discussed in order to understand the reasons behind the observed changes. The themes and conclusions drawn from the group as a whole are discussed in the next chapter. These draw from both the case studies and the belief surveys. Since the belief surveys are mentioned in this chapter as well, a brief note to provide context for the numbers reported here is necessary. Full detail on the belief survey data and its analysis is included in chapter V.

#### **A Note on Belief Survey Data**

In each of the case studies that follow, I summarize the teacher's responses on the belief survey. Chapter V has a detailed description of the belief survey results, including an analysis of the categories' reliability and the changes from the pre-test to the post-test. One significant result was that the categories had low reliability scores (using Cronbach's alpha), which indicates the items in each category were not reliably testing the same construct, thereby limiting the conclusions that can be drawn from them. Certain items were eliminated from each category in order to maximize the reliability, and the mean response scores for these adjusted categories are reported in each case study. Despite the low reliability, the pre- and post-test response scores for each individual still provide

valuable clues about their beliefs regarding student learning and are therefore reported for each teacher. The belief surveys were used to help formulate the third interview, as well. I noted the changes from the pre-test to the post-test for each teacher in order to look for clues regarding changes in beliefs. I made a note to make sure our conversation addressed those particular beliefs if any possible changes stuck out.

## **Scott**

### **Background**

Scott was participating in the second year of the program. A career-changer, he had been teaching for five years after working for several years in engineering and business. His undergraduate degree is in electrical engineering, which included a significant mathematics component, and he also earned an MBA while working in industry. After leaving his career in high-tech, he expanded an interest in working with youth into a “youth leadership business for a year, [and then] I had to make a financial choice there on that one and it was either go big and get away from the youth or change topics and I said I like the youth side of things, so I decided to start teaching.” He spent the first three years as a teacher in “traditional” public schools. At the beginning of this project, he had been teaching for two years at his current school, where he, as the math expert, team-taught math, science, and technology with a science expert. The six-year high school is organized into three divisions, and students spend two years in each division. Scott taught division two, which is the equivalent of grades nine and ten. In addition to science and technology content, during the

two years they cover “everything from algebra and geometry to introduction to trigonometry to logic, statistics.” His classroom was very informal (students called him and his co-teacher by their first names) and involved a lot of student-teacher interaction and group work. He described the teaching philosophy at his school as “student as worker, teacher as coach – so you’re really not talking head, you’re really trying to formulate things as experiential learning, and coach them along in that to the greatest degree possible.”

In fact, Scott identified the opportunity to experience that teaching philosophy from the perspective of a student as a big reason for his participation in the program:

I’m coming from a traditional math education where you’ve got, y’know, to a great degree you’ve got a talking head that shows you concepts of how things work, and then you do a bunch of them. You may play around with them, you may expand upon it, but it’s still that I didn’t have the “learn through trial and error” experience. So this was a great opportunity because here’s something proposing, like, “great, you can now go experience what you’re trying to get your students to experience and then have conversations about how do you teach that and follow on with that” [...] So it was like ... “Oh great! This is what I’m supposed to be doing at school, and I don’t have any experience in that style.”

And, indeed, Scott said the first-summer experience changed the way he approached his classroom:

T: So you found it changed the way you approached your classroom? Maybe not on a day-to-day basis, but at least at some points?

S: Yeah, if not on a daily basis.

T: So [...] what was it about last summer that you think prompted you to make those changes?

S: Well, some of it was experiencing it myself, I think. Certainly. It also, I think, has a lot to do with just really, really emphasizing the whole concept of learning through experience and making it absolutely forefront in my mind and [...] giving me a basis for changing my paradigm for looking at the world.

Scott's emphasis on taking the role of a student played a significant role in his second-summer experience, as well, and was a contributing factor in his thoughts and discussions regarding student learning. Next, I discuss his experience in the RLE program to demonstrate the motivation behind the shifts in his conceptions that shall be described.

### **Scott's Summer Learning Experience**

Scott identified the opportunity to be a learner in an experiential setting as an important reason for starting the program and a significant aspect of his participation in it. He said his experience from the first summer "just really, really emphasiz[ed] the whole concept of learning through experience and [made] it absolutely forefront in my mind" and gave him "a basis for changing my paradigm for looking at the world".

During his second summer, Scott (along with his partner Jennifer) chose linear Diophantine equations as a project topic because he had found the material to be interesting and wanted to understand more about it. He said he "understood the mechanics of 'ok this stuff comes together this way and things like that'" but that he "didn't really [...] grasp the whole thing in the big picture [...] it was still kind of mystical at the end of last year". The organizers provided some prompting questions for the project, and he noted that looking at those had encouraged him to "step back" and "just take a deep breath and look at it." Doing

so led him to the realization that “Duh – It’s an equation for a line in standard form!”, which sparked the rest of the investigation :

[...] and the little lights started going off all over the place [...] really, that little blink kind of pulled the whole things of what they are how they work, what these solutions mean, what non-solutions mean – all of a sudden I really felt it and visualized it and really grasped it.

Scott acknowledged that the prompting questions inspired some of the directions taken during their investigation and “sanctioned” other directions as worthwhile, but felt as though the bulk of the work had proceeded through independent exploration. He stressed that his learning trajectory consisting of four steps: (1) gaining experience, followed by (2) stepping back in order to generalize, and then (3) gaining a key insight (the “blink”) that allowed him to (4) translate into a new, more meaningful representation.

Social interactions played a role in Scott’s learning, as well. Working with a partner was a meaningful part of the experience for him. He noted that

it’s good to verbalize and bounce ideas off and get the synergy coming out of that. Because I go into brain freeze real easy and I can’t break out of seeing in my tunnel on my own very well. It’s difficult for me to do that, but I’m real open to and receptive to other people’s ideas.

However, while he felt like his partnership with Jennifer was a productive one, and they enjoyed working together, he did not believe that it was as “synergistic” as it might have been with other participants with whom he worked. He enjoyed working with her, but did not feel like they had gotten maximum value out of their partnership. Indeed, they mostly worked independently and then came together to share results rather than working closely with each other.

The group's mentor helped them throughout the project, providing guidance through questioning, but never telling or lecturing. She also encouraged the pair by being "genuinely excited about [the] project and [...] amazed by stuff that we would come up with." As a "coach", she filled the role that Scott envisioned for himself as a teacher – guiding, prompting, helping without telling and without the student fully knowing all that they are being asked to do. Though in his own classroom he constructed the prompting questions himself, the aspects of instructor and guiding questions were in place similar to the structure of his own classroom.

Scott's learning experience closely mirrored the experience he hoped to provide for his students. His classroom practices and conceptions of student learning emerged through the lesson planning tasks. His conceptions of student learning according the belief survey are discussed next, followed by the hypothetical learning trajectories he constructed and described.

### **Belief Inventory**

Scott's average response scores for the belief inventory are summarized below in Table 2. Categories I and III showed no significant change, indicating that his beliefs about the degree to which students construct or receive knowledge and his expectations of student capabilities remained unchanged from the beginning to the end of the summer program. Furthermore, the scores indicate that he agreed with the idea that students principally construct knowledge for themselves (category I) and that students are capable of significant independent insights (category III). His score in category II changed

from a 3.5 to a 3.875. This may indicate that he saw the processes of student learning and mathematics research as more similar at the end of the summer than he did at the beginning, when his beliefs on the matter were rather neutral.

TABLE 2: *Scott's Belief Survey Response Scores*

	Adjusted Category I	Adjusted Category II	Adjusted Category III
Pre	4.111	3.5	4.0
Post	4.0	3.875	4.0

As I will show below, the significant agreement with categories I and III is consistent with the conceptions he professed in the interview, and with the goals of the professional development program. Thus, the lack of change in these two categories is unsurprising. However, the shift in his beliefs about the relationships between students learning in mathematics and mathematics research is significant in light of the way his perspective on his beliefs changed over the course of the summer program. The sections below will detail this change and suggest some reasons for it.

### **Hypothetical Learning Trajectories**

Scott's first lesson was to focus on solving linear equations, though it grew into a more general introduction to linear equations. He did not use the sample lesson plan for the linear equations lesson. For the lesson on defining function, he consulted the sample lesson, but did not use it extensively as he discussed how he would teach the subject. Below, I will discuss the hypothetical learning trajectories he constructed for both lessons and some changes in the way he talked about student learning.

In the first interview, Scott identified the following learning goals for his lesson on linear equations (quoted as he wrote them):

- Recognize and know when a linear equation is appropriate
  - In a word problem, a real life situation, an equation, a table, or a graph
- Create a linear equation from a table, word description, or graph
- Understand three forms of equation (slope-intercept, point-slope, and standard form)
  - What the different forms mean
  - Why they are useful
  - What does each part (i.e., coefficients and constants) mean?
  - Convert all forms to slope-intercept form
- Construct graphs by hand, with a calculator and with a computer (Excel and Geometer's Sketchpad)

In the second interview, he added the goal "Solve for the coordinate pair given either  $x$  or  $y$ " but left the goals unchanged during the third interview. For his lesson on defining function, he identified the following goals, which remained consistent throughout all of the interviews:

- Realize that a linear equation and a function statement are synonymous terms
- Convert linear equation statements and vocabulary to function terminology (i.e.,  $f(x) = \dots$ )
- Identify and explain functions in real world situations
- Describe real world relationships using the language of functions.

Note here the emphasis on multiple representations and translating between "real-life" situations and mathematical concepts. Many of the verbs he used - "recognize", "understand", "construct", "convert", "identify", "describe" - as well as the specific items he used to define "understand three forms of equation", indicate a focus on conceptual understanding. Still, his learning goals were principally content-oriented. That is to say, he saw the acquisition of particular



content to be the main goal of his class, but wanted that acquisition to be deep, multi-faceted, highly connected, and meaningful to individual students.

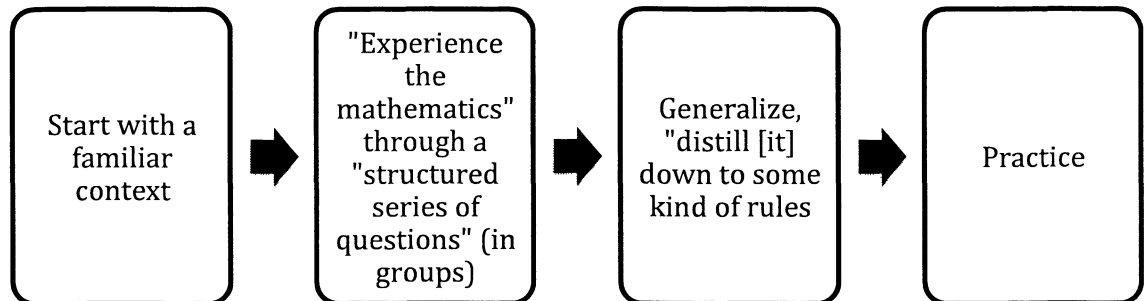
Scott also identified a very consistent lesson trajectory that guided the development of his lessons. It emerged during the first interview and remained consistent throughout every lesson he discussed during all three interviews. The lesson I observed in his classroom also followed this format. The basic structure consisted of beginning with a familiar problem context and posing a motivating problem in this context. For example, when introducing linear equations, he suggested an activity where students would time themselves running a set distance and then use that to predict how long it would take them to run other distances. He repeatedly stressed the importance of starting with a “first hand experience”, stating that it

build[s] their knowledge base [so that] they have experience that they can relate to. Because they’re participating in it, they remember it, and they know [better] how to apply it and they recognize it better than if you just tell them what to do and then they practice it. They really seem to get a much better retention and application and how to apply it [better] if they have their own tangible experience to relate to on it.

After an initial activity that tapped into their prior experience and related it to the subject at hand, a “structured series of questions”, worked on in groups, would then encourage students to “experience the mathematics” in the established context, building toward a generalization of what they had seen. In the linear equations example, these questions would encourage students to extrapolate their results and eventually construct an equation that modeled their position in time while running. They would then extend this to other contexts in order to

generalize the concept, to “distill [it] down to some kind of rules.” They would then practice using those rules. The basic structure is shown in Figure 2.

FIGURE 2: *Scott’s General Lesson Trajectory*



Scott’s learning goals and lesson structures are consistent with a model of teaching termed *content-focused with an emphasis on conceptual understanding* by Kuhs and Ball (1986) or *child’s constructed understanding and interest driven model* by Ernest (1989). In other words, content goals drive his lesson development, but they are organized around students constructing their own knowledge and are sensitive to student understanding and interest.

The consistency of his professed learning goals and of his lesson structures throughout the interviews and during the classroom observations was striking. These aspects of his basic hypothetical learning trajectory did not seem to have been significantly impacted by his participation in the mathematics immersion program. In many ways, his existing philosophy of lesson goals was consistent with the goals he experienced as a learner in the summer program – to understand deeply in multiple contexts and to develop connections between concepts. Thus, it is perhaps unsurprising that the nature of the learning goals he identified remained unchanged over the course of the summer experience. However, even though the trajectory of student learning Scott identified did not

change, the way he discussed students progressing through the lessons demonstrated that his conceptions were affected.

### **Scott's Discussion of Student Learning**

In each interview, the lesson planning task was immediately followed by a request for the teacher to discuss what a “typical” student in their class would be thinking as they participated in the lesson. During the first interview, after discussing the lesson on linear equations, Scott said he expected students to “form connections, take their experiments and take their observations and take their information and to be able to generalize that into some kind of form or some kind of rule, something like that.” This was essentially a reiteration of the trajectory he constructed when planning the lesson. It concerned his expectations of what the students should be doing, but did not delve very deeply into the means by which he expected the students to do these things. That is, he made it clear what he expected the students to do during the lesson, but not how he expected it to occur.

Responding to the same request regarding the lesson on functions, he claimed that students would respond differently, and noted the attitudes of students at different ends of the spectrum. However, the focus ultimately returned to the expected lesson structure:

I guess I don't have a good answer, because [there will be students for whom] it's like this [snaps fingers back and forth] and then there's that other end of the spectrum [where] there's going to be ones that are like “oh my god, I was just thinking I was getting that thing and now we're changing what we're calling everything and I'm not really understanding that. Oh now we've already moved on again?” [laughs] So they're still at a different place, and then there's going to be ones between those two extremes, which is part

of my challenge ... trying to cut the balance between giving the individuals what they need ... so in notes we've got to reemphasize things like 'this is important, put it in your notes'. Other ones I may need to give them things the next day that was a copy of the important notes or to have just a ... recap.

Both of these excerpts highlighted his content-oriented viewpoint during the first interview. Scott's basic trajectory and general discussions showed that he believed students construct their own knowledge and that the job of the instructor is to guide and facilitate that construction. However, in discussing how the construction might occur, Scott returned to discussing the lesson structure and only mentioned student participation in it in general terms.

During the second interview, Scott made only superficial changes to the two lessons, and his basic trajectory remained the same. When asked to talk through a typical student's learning, he said the following:

I think there will be some contingent of students that will be doing the "why are we doing this, how does it relate to math?" sort of thing [...] I think [...] that the majority of the students, though, will be trying to see the understanding below the surface of what they're doing [...] anticipating where the path is going to go or trying to figure out where the path is going, where they're going to get led.

He again made note of two different groups of students – those who willingly engage and those who do not. However, the focus of his discussion was not on the lesson as much as it was the way the students engage in the lesson. It was much more oriented toward the students' cognition. When discussing the second lesson on defining *function*, Scott identified renaming mathematical objects as the key objective. When asked about students' thinking as they participate in the lesson, he said:

I see a lot of students really wrestle with the whole fundamental concept of algebra [...] the whole renaming of things, or renaming kind of relates to the concept of substitution. At this ninth grade level, [...] they're not real, real comfortable with all of that. They seem to wrestle much more than I would have expected. [...] I mean, you know, if I say "this pen and that pen just represent pens, it's like "ok, it represents pens", but that one's black and this one's blue." [...] You know, they're getting hung up on [that sort of thing], so yeah, for something that conceptually, to me, just seems fundamental, to have somebody wrestle with it is surprising.

He again focused on student cognition, though it was in reference to his own expectations of student thinking. He spoke of them "wrestling" with the concept.

In this second interview, Scott again emphasized the ways students participate in the lesson, but his focus was oriented less on the content of the lesson and more on the learners. He was anticipating their reactions and struggles more than in the first interview, though, in the second lesson, this was again filtered through his own expectations of what they should be doing in the lesson. In the third interview, conducted during the school year, the shift from content-oriented to learner-oriented was even more dramatic. In that interview, he compared learning mathematics to learning a new language:

I do think a lot of the mathematics at this level is very much like a second language. It's got its own vocabulary terms, they have meanings, you've got to understand what they are, you've got to put them together in the right context and the right order in order to successfully communicate what you're talking about. You know, we have our own notations [...] and you've got some that look the same [as familiar letters], and some that don't look the same. And the ones that look the same may not mean the same thing as they did in the other context, and the ones that don't look the same, you've got to figure out what they mean to begin with.

The metaphor of language was an important one for our discussions during this interview because Scott returned to it repeatedly – it clearly resonated with him.

When discussing how student learning would proceed through the linear equations lesson, he characterized his students as learners of an unfamiliar language:

[At the beginning of the lesson,] they're going to be in that mode of, like, a beginning new language speaker. So, they're going to say the word, but they're going to have to translate in their mind to a different language, like "what does that mean?", and try to work through it, translating as it goes [...] They're not going to be speaking fluently on it with a deep understanding of what those terms mean and it all makes sense. They'll be working at hearing the term or using the term, but having to interpret [...] and kind of feeling fuzzy and ungrounded at it as they're working through, trying to gain those understandings.

Note the attention paid to the experience of the students in the above quote.

While he was still discussing the ways in which they participate in the lesson, the focus was less on the steps he expected them to undertake in order to master the content and more on their feelings and experience during the lesson. This altered focus remained consistent as he discussed student learning as they hypothetically engaged in the lesson on defining *function*:

It's going to be exactly the same thing I said before [about feeling uncomfortable learning a "new language"]. I think on here [introducing  $f(x)$  notation], though, because they have less experience than what they did on the  $y$  equals  $mx$  plus  $b$  thing, which they already had some foundation in, [...] but] this function vocabulary item would have been new to them, so they're wrestling much more with that. Trying to understand that there's an equivalency between a  $y=$  statement and an  $f(x)=$ , [...] A lot of students still aren't real comfortable with just an algebraic expression [...] they're still kind of wrestling "well, what's the difference between a  $y$  and an  $m$  and a  $b$ ? Those are all just letters, why do you call one a constant and one a variable? And now you're telling me that this thing I'm not real sure of is a variable has another name that has more letters in it? And some of it's a variable thing and the other thing doesn't work on anywhere in there?"

In this excerpt, he discussed student participation in the lesson, but his perspective was once again oriented toward the experience of the student rather than their actions in the lesson.

### **Discussion**

To summarize, Scott's professed beliefs about student learning did not seem to be affected by the professional development experience. If anything, they were reinforced by it. During the third interview, he described his teaching philosophy and corresponding classroom structure during the third interview as follows:

Our format should not be talking head teacher [where] you write notes and play mimeograph machine or Xerox machine and just reproduce what you were told how to do without gaining understanding of it. My job is to help you [the student] learn and discover those things. So to a great degree, from that standpoint, it very much is consistent with [the professional development program's] philosophy of "we'll give you some guidance, we'll ask you questions, but get you to learn through the discovery." And by doing that, your retention and understanding of it, I think, is greater than if you just hear something and you can parrot it back without ever really internalizing it and manipulating it. So it's learning through experience, drawing on your experience to come to conclusions, verifying the conclusions and then using the conclusions as the launch point into something new.

His characterization of the summer program and of his own classroom were consistent with my observations of both and with his descriptions of his classroom and teaching throughout all of our interviews. The consistency of his hypothetical learning trajectories and accompanying statements indicates a central belief in individual construction of knowledge. In fact, this conception of learning seemed so central to his attitude and approach toward learning that it could be considered as knowledge to him. Similarly, the professional

development program was organized around the work of a mathematician and focused principally on individual cognition and action, and the long-term research project only reinforced these principles. Thus, it is perhaps unsurprising that Scott's conceptions of student learning did not show a great deal of change.

However, despite the lack of evidence for changes in Scott's conceptions of student learning, the way he talked about that learning shifted a great deal. His perspective on the student learning process shifted from principally content-oriented (focusing on the students as respondents to the content-driven lesson) to more learner-oriented (focusing on the individual thoughts and experiences of the students). Part of the much more significant focus on students during the third interview may be attributable to the context of the classroom, a context from which he was removed during the RLE program. However, I argue the beginning of the shift in perspective was seen in the second interview, and that the digestion of the summer experience coupled with a return to the classroom developed it further. Thus, the roots of the change in perspective are likely to be found in Scott's summer experience.

Indeed, Scott's learning experience closely mirrored that which he desired for his students, and he noted during the interviews what an important experience this was to him. It seems this student-in-experiential-learning experience led to his shift in perspective. In other words, instead of drawing parallels between research and learning, he was drawing parallels between his own learning and the learning of his students, leading to his increasingly empathetic perspective on student learning. Furthermore, I argue that since he



did not form meaningful conceptions about the nature of mathematics research, the shift in his category II score on the belief survey was the impact of experiencing learning in a setting that had been labeled “research-like”. Since that experience resonated with his conceptions of student learning, he correspondingly adjusted his responses to the items in category II. Both of these claims are supported by a lack of robust conceptions regarding the nature of mathematics research.

### **Conceptions of Mathematics Research**

As mentioned in the Methodology, the second and third interviews included a question about the nature of mathematics research, posed in the form of a hypothetical query from an interested student. In the second interview, Scott was asked, “if a student asked you what mathematics research was all about, how would you answer?” He responded as follows:

That’s a good one. I haven’t ever thought about actually answering that. [laughter] Um, I guess that I would have to relate back to what our experience is here and answer them like, “well, research is going and working with real tangible examples to get a feel for the dynamics of what’s actually going on and then determining if you can generalize – see if you can’t find patterns to generalize that, or patterns within patterns to generalize your experience and then go about determining if you can substantiate that or prove that is the case in general.”

He referred to his summer experience, using language that closely mimicked that used by the program organizers (“examples to get a feel”, “find patterns”, “generalize your experience”). In fact, when asked how he arrived at that particular answer, he responded by saying that “it’s the whole [summer program] experience of just doing exactly that! [...] I believe that you’re kind of doing the

math research, even if it isn't leading edge stuff." Thus, while he did describe some aspects of mathematics research that were consistent with the descriptions reviewed earlier (cf. Muir, 1996), he mostly parroted the descriptions that were contained in the program.

During the third interview, he initially addressed the question by referring to a research project about which he had recently read that concerned "the mathematics of pancake flipping", which he planned on using as part of an extension activity for his advanced students. He described it as

current mathematical research going on right now, today. And they don't have an answer for it yet. But they have discovered an application for it and are currently using it in an area of science that they weren't even aware of it when they started playing with the underlying mathematical concepts which they'd played on for decades [...] The number of flips that you need to get the pancakes in a proper order directly relates to the number of genetic modifications in your DNA coding that separates you in evolutionary steps.

He repeatedly used the pronoun "they" to refer to those engaged in mathematical research, as though it is undertaken only by others – some set-apart class of individuals. He clearly did not see himself as a participant in it, and he seemed to regard the process of mathematics research as something to which he and his students only had partial access. Furthermore, he was focused on the results and connections of a particular research program rather than on the process by which it is undertaken. That is, the access that he and his students had was to the product of mathematics research – they were observers and consumers of it, not participants in it. Indeed, when pressed to describe the "essence of mathematics research", he said "I guess I hadn't really thought about the

question before. I guess the essence of what makes it mathematical research is that you are taking a problem and trying to find ways to mathematically model it.” Here, he admitted to not having a very fully formed conception of mathematics research, and offered a rather trite and naïve summary.

The lack of well-formed conceptions of mathematics research was further evident when Scott was using the “new language learner” metaphor for learning mathematics. That led to the following exchange:

**Scott:** you will never be a fluent speaker of Spanish if all you’ve ever done is gone to Spanish class and listened to Spanish on tapes as you drive back and forth. You may technically have a lot of the words, but you’re not going to be fluent at it [...] you’re always going to be a Spanish translator. To get to be fluent in it and become a Spanish thinker, you need to go immerse yourself and live in a Spanish culture. So it isn’t just a matter of just practice, you also need to experience what does that mean and what is that? [...]

**Todd:** So, do you see your classroom as a place where students are immersed in mathematical culture the same way a Spanish learner might be immersed in Spanish culture?

**S:** Try to be. I’m not sure how successful I am, but I try to be [...] giving the opportunities for having the experience [...]

**T:** So where do you think a true experience of mathematical culture happens?

**S:** The matrix! [laughter] ... You know, I don’t know. To be honest about it, I’m not sure what that would look like or what it would be.

Despite his participation in a program designed to stress the culture and nature of mathematics research, Scott did not form very robust or meaningful conceptions of that process. Thus, it is apparent that the changes in his perspective toward his conceptions of student learning were not the result of

drawing parallels between student learning and mathematics research. Indeed, in the third interview, he was asked to respond verbally to the belief survey item *when mathematicians “do mathematics”, they are doing something fundamentally different than when students “do mathematics”*. On the first survey, he circled response 2 (Disagree). But on the second survey, he circled response 3 (Neutral), and during the interview, he agreed with the statement, claiming that most students, when thinking about mathematics, are really

thinking of arithmetic, or something closely related to arithmetic.” Mathematicians, however, “are really dealing closer in concept to philosophy than arithmetic. Because they’re looking at the bigger picture and things in general. They’re trying to go from specific observations to general forms [...] and I don’t think most students do that – they’re still lost at [the question of] what are the specific manipulations to make the numerical experience happen?”

Though this indicates he at least considered comparisons between the processes of mathematics research and student learning, his participation in the program does not seem to have prompted him to draw parallels between the two.

It seems his referent for mathematics research was two-fold: inaccessible work done by “others” and his own experience during the summer, which he was told was “research-like”, though his discussion indicates that he did not consider it to be genuine research. Given his poorly-developed definitions for mathematics research, the shift in his response scores for category II on the belief survey is surprising. However, the interviews reveal that the change occurred not because he was learning what mathematics research was all about, but instead because his experience as a learner was contained within a project he was told was a facsimile of the mathematics research process. The

opportunity to experience learning mathematics as a student was extremely impactful for Scott, and, given his lack of conceptions about mathematics research, seems to have been the dominant factor in the observed changes. Because the setting of that experience was described to him as “research-like”, his beliefs about the relationship between research and learning shifted accordingly.

### **Conclusion**

Scott's primary beliefs regarding the nature of student learning were not changed, perhaps because of their initial consistency with the philosophy of the program, but his perspective on their experience as learners did. He demonstrated increased empathy for their feelings and experience. Furthermore, the changes mirrored his own experience as a learner in the RLE program as he participated in an environment that he saw as similar to his own classroom. His conceptions of mathematics research remained ill-formed, leading to the conclusion that parallels between research and learning were not the primary motivator behind the observed changes. He did note some parallels between the two processes, but these were constructed through the intermediary of his own learning experience in a setting described to him as “research-like”. Ultimately, Scott was drawing parallels between his own learning and that which he expected of his students.

## Jennifer

### Background

Jennifer was participating in the RLE program for the second summer. She had fourteen years of teaching experience, ten in private schools followed by four in public schools. After graduating from college with an undergraduate degree in engineering, she secured a position at a private high school teaching English as a Second Language. After a year, she moved into teaching math and physics before eventually moving on to other private high schools as a mathematics teacher. Her position at the time of this research project was teaching mathematics in a public, urban high school. She made the move from private school teaching because she “wanted to see what the public school world was like.” She described teaching in private schools as “awesome” but worried that “there [was] no real education background at all between any of us.” She noted significant differences between private and public school students, characterizing private school students as “really want[ing] to learn ... it’s not that whole babysitting classroom management thing.” Thus, she found her current position to be more challenging, but said that those challenges had helped her learn “a ton about teaching and different methods [for teaching]” even though “the teaching part is not nearly as enjoyable as it was in the private school.” She taught two different levels of precalculus (honors and College Prep 2, or CP2), algebra, and geometry, and was the department “leader” for geometry. I was able to observe two precalculus (CP2) classes when I visited for the third interview.

Jennifer disagreed with the style and philosophy of many of her colleagues. Her department had a standard curriculum and gave common midterms and finals, but, feeling that important content was removed, she often augmented the curriculum with additional activities and lessons. As a result, she was often “a little behind” other classes, but claimed to catch up by the end of the semester: “while they’re spending a ton of time reviewing for the exam, I feel like I’ve been reviewing all along, so I don’t have to spend as much time.” In addition to adding content, she also felt that she challenged her students more than other teachers in her school. Students who transferred into her classes from other teachers often earned lower scores on quizzes covering the same content, and described the other teachers’ quiz as “much easier”. She felt that the other teachers were not going “in depth”, but felt validated because her students’ common exam scores were ultimately higher than those of other teachers, and the calculus teacher reported that calculus students who had Jennifer for precalculus were better prepared and outperformed other students in calculus. It was important to her that students in her classroom be challenged and pushed to understand concepts deeply.

In order to renew her public school certification, she “needed to get a master’s degree”, and the opportunity to do so was part of what attracted her to the RLE program. Though she did not start the program as a graduate student, she applied and was accepted to a master’s program at the university during her second summer. Despite that fact that she had a lengthy commute to and from the program (described as “really a pain”), making the logistics of her

participation difficult, she said, “I really like the program and I felt like I gained a ton and definitely felt like I was a better teacher because of it.” After the first summer, she was hesitant to incorporate the “exploration” style in her own classroom, but attempted to incorporate some ideas into her algebra classes (because the book was “atrocious”). She reported the following experience:

I’m always appalled by [students’] inability to do basic arithmetic and then just have no connections, no understanding. I feel like they really have been taught “you just do this, you just do this” and they don’t really think about it, you know? So how can I get that [understanding] into my classroom? So [... the organizers] are saying [this style] is going to help with problem solving and stuff. And then after my first year I was just not convinced at all ... I mean, I totally understood it, I totally got a good feel for what was going on and stuff, but I don’t know that that really developed problem solving. Then I started kind of using kind of that discovery style and kind of making connections for them in kind of little baby problem sets kind of things. I did a little bit with [precalculus students] but really spent a lot of time with the algebra two kids. And I was amazed, like, it totally did!

At the beginning of the semester, the students “couldn’t solve a one-variable equation – they were making mistakes as to when to add, when to subtract, when to multiply, when to divide.” However, she worked to provide opportunities for students to solve problems independently in order to develop skills and focused on justification and connection between topics. On the final exam, she purposely avoided reviewing a problem that was going to be unfamiliar to her students in order to observe their reactions. As she described it:

They all answered it and I would say, like, 90 percent of them got it completely correct. See they weren’t afraid because they had been used to looking at things they hadn’t seen and just kind of seeing. I think part of it was confidence and part of it was they sort of understood kind of the background stuff, so that they were open to solving it.



Thus, the first summer had initiated some change in Jennifer's practice, and she viewed these changes as having positive results for her students. I will now discuss her experience during the second summer in order to illuminate how her experience with mathematics research impacted the way she conceived of student learning.

### **Jennifer's Summer Experience**

Jennifer worked with Scott on a research project investigating linear Diophantine equations. Between her commute, the number theory coursework, the research project, and the lesson-planning project, Jennifer felt "pulled in so many directions" that it was difficult to completely invest in the project. The fact that she did feel so busy contributed to her choice of subject area: "I thought that being [...] a topic that was familiar may help." She concluded, though, that "it probably would have actually probably been better had it not been [familiar]. It was almost like [...] you have to go further to get anything new." Furthermore, the open-ended nature of the project was difficult for her initially:

I didn't have a whole lot of comfort in just exploration, so it was definitely hard at first to just kind of play around with no direction, sort of. I mean, they gave you some questions or whatever, but they're pretty open, pretty vague. So, yeah, at first it was just like, "I don't know what I'm doing with this."

Her own past experience as a learner, even from the first summer, left her uncomfortable with unguided exploration. She said, "I've always been very successful with just kind of being taught [...] I don't think I really had any experience [with exploration ...] everything was just so formally taught." Because

of her previous learning experiences, the lack of structure increased the discomfort she felt with her project.

Ultimately, she moved forward with “a lot of help.” She drew upon the group’s mentor and her partnership with Scott, noting that it would “be really hard to do it on your own.” Thus, social processes, in particular careful guidance by an expert (the group’s mentor), played an important role in increasing her comfort level. The mentor played the role of guide rather than teacher, responding to directions and ideas suggested by Jennifer and Scott rather than proposing them herself. Jennifer viewed her partnership with Scott as beneficial, though she noted, “at times, [Scott] wanted to go off in this direction, I wanted to go off in this [other] direction, so we just went off in different directions, and they obviously came back [together] anyway.” She appreciated the opportunity to access additional perspectives, particularly that of a skilled expert, but did not work extremely closely with her partner. They functioned more as cooperative, but independent, explorers working as two individuals rather than as one entity. Nevertheless, she identified the influence of others as key to her progress over the course of the project.

By the end of the summer, Jennifer said, “I could do some on my own and feel pretty comfortable, like, making new connections, and I feel like I was, but I feel like at that point we already had channeled in what we were looking for.” That is, she felt comfortable exploring in a more precisely defined space, but the discomfort with initial, “wide-open” problem-solving remained: “it was almost overwhelming, like there’s just too much, there’s no way we can figure this out.”

In fact, that open-ended exploration was the key difference between her first and second summer experiences:

Last year, you have the problem sets and they have the little mini explorations and stuff. Like you, I don't know, you might spend like a half hour looking at something and being like "yeah, I don't see where this is going". (laughs) So, I think, [this summer's project] definitely just, it sort of forced you to keep going when you were like "yeah, I don't see anything." And definitely, there's just more comfort where if I had a problem I would just feel free to just go ahead and dig in and see where it takes me and feel comfortable with that. Before I'd be like "yeah, I don't see the point." And stop.

The research project required her to push through confusion and frustration associated with an apparently fruitless search for understanding. During the first summer, if a problem or exploration proved too challenging, moving on to something else was an option. However, in the project, she had no choice but to continue working. Ultimately, she felt more comfortable and empowered in situations where she did not understand the meaning or "the point".

Indeed, removing any options other than self-reliance led to one of Jennifer's significant breakthroughs over the summer. She spent part of her commute on a commuter train, and sometimes used that time to work on mathematics, and it was in that setting that she first completed a proof "all by [her]self". She noted that the circumstances on the train created a good environment for such breakthroughs:

I don't think it's the train, I think it's because I listen to music and I can't get help, so I have to do it on my own. Whereas if I'm [at the university] I'm just like 'yeah, I don't get this.' (laughs) And it's like the first one I actually did completely by myself was because I couldn't get help – I was on my own.

Thus, just as with the research project, Jennifer felt that she benefitted from having her supports temporarily removed. She had felt unsure that she would be able to complete a proof without any outside consultation, and experienced a tremendous sense of accomplishment when she did. Jennifer desired to solve problems and master the material on her own, with minimal support from experts or other outside sources. She and her tablemates, with whom she worked closely on the number theory problem sets, intentionally avoided counselors who would give too much away.

Jennifer's desire to come to understanding through her own processes and at her own pace was related to her dissatisfaction with assumptions in proofs and problem solving. She enjoyed working with her tablemates because, as she stated it: "none of us are ok with knowing something's true, like, in a proof [...] you can't [write] 'detail, detail, detail' and then be like 'oh, we know that's true.' And just kind of skip." She wanted every step to be fully developed and proven. That desire to avoid any gaps and for full development of concepts extended to her teaching, which I shall discuss below. This, along with her experience with long-term exploration, were the two main themes that emerged from her summer experience that were echoed in her teaching, as she described:

In the past [in my class] I didn't necessarily do exploration, but maybe a little bit of asking why. Like why things happen, why is that true, is that always true, that sort of thing. That's' definitely played a role in the way that I teach and the way that I explain things. Because I'm writing down things, I'm constantly thinking, like, "well why is that true?" So then when I teach it, I kind of explain why that's true and I feel like in doing that, it definitely makes connections clearer. And so, I'm always, as I'm thinking about lesson plans or, you know, thinking about how I'm going to teach things, those questions had never come up [...] Maybe on

occasion, I was doing something a little bit new to me, but things that I've done for years, I'd never really thought about it kind of from that direction – like, why would they come up with this method, or why would they want to do this. Kind of just trying to understand the reasoning behind even just methods of doing things. So that's definitely a change in approach.

She claimed that her desire to understand the motivation and reasoning behind problems or concepts had changed the way she taught by increasing her awareness of these issues for her students and prompting her to question the motivation behind choices made in curriculum materials.

Due to being overwhelmed with responsibilities during the summer, Jennifer did not begin to think about how the experience might impact her teaching until “the last week” of the program. She contrasted this to the first summer, when she had “thought about it a lot”. Furthermore, her master’s program required her to take an abstract algebra class during the fall semester, which, along with her teaching responsibilities, consumed her time and attention. Thus, she believed that she had not had time to properly process her summer experience and she had not tried as many new things in her classroom as she would have liked. Nevertheless, changes in her approach betrayed shifts in the way she conceived of the student learning process. I turn now to discussing those conceptions, drawing on Jennifer’s summer learning experience to explore and explain the roots of changes in them.

### **Belief Surveys**

Jennifer’s response scores for both administrations of the belief survey are shown below in Table 3. Her scores both before and after the RLE summer program indicated that she believed students constructed their own knowledge

rather than received it (category I), that mathematics research and student learning were somewhat similar processes (category II), and that students were independently capable of significant insights (category III). However, her agreement with each of those beliefs was mild. All of her scores fell between 3 (Neutral) and 4 (Agree). Thus, these results, suggest that Jennifer did not hold strong beliefs in any of these categories.

TABLE 3: *Jennifer's Belief Survey Response Scores*

	Adjusted Category I	Adjusted Category II	Adjusted Category III
Pre	3.667	3.875	3.625
Post	3.1667	3.625	3.4

Furthermore, though all of her scores changed from the beginning of the summer to the end, none changed significantly. The largest change was in her category I score, indicating that, at the end of the summer, she agreed less with the idea that students construct their own knowledge than she did at the beginning. However, the difference between the scores is not large enough to draw that conclusion, thus it is appropriate to turn to the interviews for evidence to either support or deny the significance of the difference in the scores. I will first examine the Hypothetical Learning Trajectories that Jennifer constructed during our interviews, then her conceptions of mathematics research, highlighting how her summer learning experience played a role in shaping each.

### **Jennifer's Lesson Trajectories**

The lessons Jennifer planned during the interviews focused on, first, solving systems of linear equations and, second, defining and determining

*domain* and *range*<sup>21</sup>. In the first interview, she described a “typical” day in her classroom as follows:

I usually go over homework [which was assigned daily] and then I usually have them do some work on some sort of remediation thing, on things they’ve done before [...] I’ll give them some sort of worksheet or we’ll do review games [using the SmartBoard, which was used extensively]. I try to do more classwork at the beginning of class and more lecture [...] at the end. And then, I have them do problem sets [...] so I review the homework or review an assignment and then teach new stuff.

Thus, her classroom typically included a brief classwork review session, either on the SmartBoard or on a worksheet, followed by new material first presented through lecture and teacher-led whole-class discussions, then reinforced with problem sets. When I observed her classroom, the lessons followed the general structure she described. The lessons she created during the interviews focused on the second part of that structure – the means by which she taught new material.

The lesson on systems of linear equations began with graphing several examples in order to “try to get a visual first” for all three cases: one solution, no solution, and infinitely many solutions. As she stated, “I think it’s important they get the feel for what the picture looks like so they kind of get an image in their head.” An added benefit of beginning this way was that that graphing would be a review of previous material and therefore built upon students’ prior knowledge,

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<sup>21</sup> Like the other interview subjects, Jennifer’s second lesson started out as defining the term *function*. However, as she constructed the lesson, she changed its focus. She felt more comfortable discussing a lesson on *domain* and *range*, and so the lesson took on that theme.

allowing her to “make the connection back to just graphing a line.” Students would graph several examples on their own to start “feeling more comfortable” with graphing and the interactions between two lines in a system. She said, “the goal is to get them to get those three different cases, and find how, if you just look at them, how do you know?” She wanted the students, with some guidance from the teacher, to build intuition about relationships between lines through repeated examples. While students did this, she wanted to draw on the definition of a solution of a linear equation in order to motivate the idea of a solution for a linear system, and then “give them kind of the overview – you know, these are different ways of solving things, we’re going to do this today, this tomorrow, this the next day and the next day.” She believed this preview of coming attractions to be “helpful” because “it’s almost like they want to be able to do things that they’re not ready to do yet [...] if there’s something that they’re not able to do, they’ll totally want to prove me wrong.” In other words, students were motivated to attempt and master content that hadn’t been covered in response to being told they were not ready for it yet.

After solving systems by graphing, she wanted to motivate other solution methods through examples that have non-integer solutions, making the solutions difficult to find without algebraic solutions methods. She thought that it was “important that they [students] understand why there are different methods instead of just ‘well I should do it this way or this way or this way’ but why would we choose this method over that method?” It was important to her that students understand the reasons for various methods and why one might choose one



solution method over another. After demonstrating that graphing is not always an optimal solution method, she proposed to demonstrate the substitution method on the SmartBoard, and then have students practice on increasingly sophisticated examples. Similarly, she would lead a discussion demonstrating the elimination method and, again, have students solve increasingly sophisticated examples.

Throughout the lesson, she wanted to lead discussions by asking students questions and modeling the types of questions they should be asking themselves as they solved problems, thus encouraging students to justify their choices. Making choices, and the criteria one might use to make them, was a significant theme of her instruction: “that’s kind of a common theme, just to try to make decisions, you know? I always try to get them to think, ‘why would they do this over this, or can I do both,’ you know?” She wanted students to not only understand the various procedures for solving systems of linear equations, but to understand the reasons they might be applied and to be equipped to decide which ones to apply. In order to emphasize the connections between methods, she required students to always write their solutions as ordered pairs, saying “I want them to always kind of go back to that graph so they always see that visual because I feel like it’s a good tie-in to know exactly what they’re doing.” She ended the lesson with a unit test that allowed students to choose their own solution methods, again emphasizing decision-making and critical thinking.

In the second interview, she maintained the same basic lesson trajectory. She did decide to include some graphing examples where the equations were

not in slope-intercept form, expecting that students could handle more complex graphing problems. Otherwise, the lesson stayed consistent in the second and third interviews, indicating that Jennifer felt strongly about its structure and believed in the learning trajectory implicit in the lesson.

Though not the same, her lesson on *domain* and *range* exhibited many similarities. In general, she said she

usually tried to make them understand something from their daily life, and with these guys, a lot of times, the focus needs to be on what the independent and dependent variables are and then we do a bunch of examples of those.

Similar to the first lesson, she began by appealing to something familiar. In the first lesson, it was mathematically familiar – graphing – but in the second lesson she chose to begin with a familiar context. A discussion would be led from there, focusing on “what are the possibilities for our independent variable? [...] and then each of those have an assignment to them, then that’s the domain, that’s the range.” Multiple examples would then be utilized for the purpose of letting students practice with the concept. First, graphical representations of functions (here, again, Jennifer underscored the importance of visual connections for her students) would be investigated. She liked to have a stick figure walk along the  $x$ -axis, and discussed the domain as those places on the axis where the stick-man could look either up or down and see the function. Eventually, they would move on to finding the implied domain from functions defined via formulas, and a class discussion would serve as the initial motivation. Though she expected that students could find the domain for any function given any representation of it, she

limited the determination of range to graphical representations, or to functions where students could find the graph and determine the range from it.

In the second interview, Jennifer kept the structure of the lesson mostly the same. However, she wanted to “do a little more exploration” in order to lead students to generalize some facts about the domain. For instance, she suggested they could “do a whole bunch of polynomials and ask them what do they notice about all these polynomial functions, what’s the domain?” She emphasized more exploration as an important route to understanding. In the third interview, she expressed dissatisfaction with the lesson as a whole, feeling that it did not “flow”, and wasn’t really allowing students to explore the concept. She did not lay out a lesson carefully, but emphasized that she wanted students to “get comfortable” with the definitions through exploration and more examples. “On the SmartBoard,” she said, referencing what she would do with the lesson, “there’s just like graph after graph after graph [...] I tell them what the definition is and we talk about, you know, a bunch, we just do a whole bunch of examples. Just getting them comfortable with finding the domain.” Even though this was what she envisioned doing in class, she did not believe that students really “make the connection” between the real-life situations and the meaning of the domain and range. She said, “I guess maybe I should do a better job of connecting it.”

In general, her lessons followed a pattern that started with motivating the concept from something familiar (either content or context), continued with a teacher-led, discussion-based lecture, and concluded with multiple examples

designed to help students notice patterns and form connections. Those examples would be some combination of individual or group work, teacher-led, whole-class discussions, and game-like activities. The lesson on solving systems of linear equations consisted of several cycles of that pattern, while the lesson on *domain* and *range* was just one.

In both lessons, Jennifer emphasized motivating concepts from preexisting knowledge, exploration, connections, and critical thinking. Her discussions of the lessons contained more references to exploration, connections, and critical thinking with each interview, though the basic trajectories of the lessons did not change significantly. Her dissatisfaction with the second lesson caused her to reconsider it during the third interview, though she was unsure exactly how she might change it. Just as explorations, connections, and understanding each step deeply were important aspects of her own learning, they were important aspects of the lessons she constructed for her students, and only became more so over the course of the summer. Her summer learning experience also impacted the way she discussed student learning, which I shall discuss below.

### **Jennifer's Conceptions of Student Learning**

Jennifer emphasized exploration, connections, and justification in her lesson planning, and her discussions of student learning were also influenced by these ideas. The pattern of her lessons indicates a belief that knowledge development is an individualized pursuit, but that close teacher supervision and direction of individuals' progress was preferable. Similar to her own discomfort

with open-ended exploration, she seemed uncomfortable leaving students to their own devices. In fact, in the first interview, she discussed student learning as proceeding sequentially through the lesson. For example, when talking about how a “typical” student might be thinking as he participated in the *domain* and *range* lesson, she “hoped” that starting with a real world application would contextualize the mathematics, preventing it from being isolated from other knowledge:

The real world application will kind of give them a reference, like, into their own life, so that they kind of understand where the math comes from. So when we talk about independent and dependent variables they kind of understand what that means [...] not just in terms of math terms or variables, but something that they totally understand. And then, with the domain and range, I’m hoping that they understand from the real world application, like, what does domain mean? What does range mean? In terms of that problem, so just giving them reference back to it.

After discussing the domain and range in a meaningful context, she said that they would be “making the jump kind of into the math world.” She then believed that experience with multiple examples would lead to understanding:

We’re going to use some symbols, functions that they’ve seen or not seen [...] and just giving a bunch of examples so they get, like, or see all the sort of different cases and kind of get to understand that and then be able to understand ones we haven’t done.

She believed that, by considering multiple examples and different cases, students would come to understand the material. She was unclear about the mental processes involved in forming that understanding. That is, she viewed the lesson trajectory and students learning trajectories as one and the same. There are two implications to this: First, she based her lesson closely on her beliefs about student learning, so the lesson structure modeled those beliefs and

she considered student learning when structuring her lessons. Secondly, though student thinking was important to her practice, she expected learners to respond in predictable ways to teacher-provided stimuli. There was little room in her lesson for idiosyncratic thinking, despite the fact that she considered learning to be an individual pursuit. Furthermore, she seemed to attribute the learning as a property of the activity itself (Heinz, et al., 2000), without significant attention to the cognition taking place.

During the second interview, the student learning trajectories she identified stayed mostly consistent, but her perspective was more student-centered, with more attention paid to students' cognition as they participated in the lesson. She said, "I'm hoping at the beginning, he [some typical student] is making connections to his own experiences, like just outside of math class," and that, 'I find that with the domain and range, especially finding it from a graph, at first, there's a lot of discomfort with that.'" She explained that she thought this was "because they're so used to being like, 'I do this and then I do this and then I do this.'" So, her expectations of students remained consistent, but she seemed to be more sensitive to the students' experience and the reasons behind it. Her discussion of students moving through the remainder of the lesson showed similar attention to student experience:

Probably, the first time he sees this [finding the domain from the graph], he's probably very uncomfortable with it. The first example, like, "I don't get it". You know, probably two or three examples in, he'll probably start to see [...] So I think it would probably take a couple of examples [for the student] to even follow and probably a couple more to be ok with finding it on his own. [...] And then I think that probably the average kid's probably pretty comfortable now with looking at the graph and getting the domain and range, but

looking at a function, probably still pretty uncomfortable with that and probably like “ok, I’ll plot these points and graph it.” I think after doing a couple of each, [he] starts to see the value in recognizing what it’s going to look like from the function.

Thus, she envisioned the student proceeding through the lesson in the same way that she had in the first interview, but she paid much more attention to his experience of it – the way he would feel “uncomfortable”, fall back on something familiar, and then “start to see the value” in new ways of doing things.

In the third interview, she again confirmed the general trajectory of student learning, and again emphasized how students would experience the lesson:

I think at the beginning, it’s really fuzzy [...] and I think as they see more and more examples, they start to get it. And they think they get it, and when they do it on their own, I still think they get stuck on some [...] And even, for some kids, even when they leave, it’s still a little fuzzy, they’re still not confident. [...] So usually it takes a couple days of kind of doing the same sort of thing – do more practice.

Students would feel “fuzzy”, and “practice” was the instrument by which their understanding moved forward.

Her discussions of student learning during the first lesson were similar. In the first interview, it was principally a rehash of the lesson trajectory: “I guess I’m hoping that they see that [...] they’re just in different forms, and hoping that they feel comfortable that it’s not a big deal to rewrite this equation.” By the third interview, the discussion focused more on their cognition and experience:

I think initially [...] there is some confusion. Like, they’re not used to substituting a polynomial into a polynomial. So usually, the first example, they’re a little iffy on and I know, like, there’s a couple of kids you can see, kind of the red alert: “what are you doing!?” But, usually after a couple of examples, they’re like “oh, that’s not a big deal. You just put that into that and solve.”

Over time, with multiple examples, she believed “it becomes more familiar, and it’s not a big deal.” In general, though, the emphasis of her discussion was on the student’s experience rather than just his participation in the lesson.

In general, Jennifer’s beliefs about student learning followed what Ernest (1989) termed the *child’s constructed understanding model*. Her instruction was based on students constructing their own knowledge, but she did not consider students’ autonomy in that construction. That is, she constructed her lessons so that the teacher directed their learning, not because she thought students incapable of directing their own learning, but because she believed that it was the optimal arrangement. Her beliefs about the course of student learning and the ways in which it proceeded did not change, but her perspective on those beliefs shifted in a manner similar to Scott’s – in accordance with her own experience as a learner over the course of the RLE program. As she experienced learning through exploration, which included feeling unsure and ungrounded, her empathy for her students increased, leading to a shift in her perspective regarding student beliefs. She increased her use of exploration activities in order to “form connections” in response to her own learning through such activities, though, to be sure, she incorporated some of this before. I shall next discuss Jennifer’s understanding of the nature of mathematics in order to illustrate that her experience led her to construct parallels between her own learning, the mathematics research process, and the student learning process.



### **Jennifer's Conceptions About the Nature of Mathematics Research**

When asked how she would respond to a student asking what mathematics research is, she reiterated the definition given by the organizers of the RLE program, but was very unsure of how to define it:

I guess I would say it's looking for a lot of patterns, looking for a lot of structure, looking for how ideas connect ... I don't know how I would really answer it. But, I mean, it seems that that's what they do. It's like this big problem that doesn't really have finality. Like it just keeps going on and on and on, you know? It seems like things have connected that you wouldn't think are connected and, then it's understanding why they're connected.

She acknowledged that "the exploration part of it" was something she would not have "understood as well" before participating in the RLE program, and she identified exploration as an important and influential part of her own learning. However, her conception of mathematics research was mostly based on what she had been told by the organizers. In the third interview, in response to the same question, she said something similar:

I always go back to that they [mathematicians] look for some sort of patterns, they look for general rules and they see if they apply to everything or if there's some sort of exceptions to the rule or whatever. And then use those rules to help them gain further insight.

So, again, looking for patterns was the defining characteristic of mathematics research in her view.

Though her definition was based in the descriptions of the program organizers, she was also personally excited about mathematics. Her excitement at her learning led her to believe that "everything in nature has some sort of

mathematical concept behind it.” Furthermore, she believed that mathematicians and students were doing essentially the same thing:

I mean, like, the students are doing mathematics and, you know, obviously at a lower level than, say, some super mathematician or whatever, but they’re going through the same process. They should be thinking about it, reasoning, to try to figure out the patterns, make connections.

However, she also said that the thought processes involved in mathematics research and student learning are “probably not quite the same “because high school kids are being guided a lot more than, say someone doing research on their own. [Someone] doing research on their own, right, they’re allowed to go in whatever avenue they want to look in, and these guys aren’t.”

Her comments on mathematics research were consistent with her beliefs about student learning. She believed exploration and pattern recognition to be key to an individual’s development of understanding, whether that individual was a researcher or a student, but she did not view students as responsible for their own learning. That is, she did not believe a student directed his or her own learning. In her own research experience, she was most uncomfortable with open-ended, broad questions that provided no clear directions, and felt more comfortable once the space to explore was better defined. She drew parallels between mathematics research and student learning accordingly, believing them similar except that students need more structure. Similarly, her own learning over the course of the summer shaped the way she discussed and thought about her students’ learning.

## **Discussion and Conclusion**

Jennifer's experience in the summer RLE program highlighted, for her, several aspects of mathematics research and learning. First, she pointed out the discomfort that can come with open-ended exploration. She actually pointed out that her experience with feeling confusion and frustration impacted the way she responded to her students' frustration:

The only good thing is I am totally empathetic with them. You know, like sometimes you're teaching and they're like "I don't get it." And you're just like, 'just frickin' think about it!' (laughs) [...] So that's been helpful because I totally understand their viewpoint on being completely lost [...] So I try to define things that I know are new to them, over and over again and talk about what they are [...] because I totally understand now what they're talking about.

So, her experience as a student increased her empathy toward her students and prompted her to be more sensitive to students as she introduced her lessons. Furthermore, the exploration in and of itself impacted her summer experience and thus her practice. As described above, she felt that the opportunity to explore problems had been beneficial to her learning, and opportunities that restricted her access to outside expertise and forced her into self-reliance had been particularly empowering. As a result of her first summer experience, she had incorporated more exploratory activities into her algebra 2 course. Though she wasn't teaching that course in the year following her second summer, she had, by the end of the second summer, decided to focus on adding similar elements to her geometry course, which I did not have the chance to observe. However, during the third interview she reported on the results of trying to increase the amount of exploration in that course:

But the summer did influence – because last year I used it more for algebra 2 [...] kind of using more exploration, kind of using that sort of ‘teach them how to be mathematicians’ sort of thing. But this year I have done more of it with my geometry [...] I have found that they have picked up on concepts better [...] I keep bringing back making conjectures and finding counterexamples, and I keep trying to incorporate that throughout, whereas before we did the lesson on it and I would let it go and not talk about it again. I feel like I’m doing a better job of letting them kind of find patterns and figure things out on their own.

Thus, her experience in the RLE program prompted her to include more exploration and pattern-sniffing in her teaching, and she believed this to be an effective change. She believed that concepts often don’t “sink in until they [the students] were asked to kind of figure I out, not quite on their own, but kind of on their own.” So, she developed an appreciation for exploration as a valuable way of developing knowledge, but still saw it to be most effective when guided a bit.

Finally, she expressed her own dissatisfaction over the summer with details or proofs that were left unexplained (by her and her group or in the lectures. She claimed that she really developed a need for everything to be logically justified. Like the above, this also extended to her practice:

In the past, I didn’t necessarily do exploration, but maybe a little bit in kind of asking “why?” Like why things happen, like why is that true, is that always true, like that sort of thing [...] I’m constantly thinking, like, “well, why is that true?” So then when I teach it I kind of explain why that’s true and I feel like in doing that it definitely makes connections clearer [...] Because I’m asking those questions of myself, I break down lessons differently or even more in trying to make connections to why things are true. And I think I definitely do a better job with connecting topics through that sort of questioning sort of thing.

During her lessons, she tried to model the questioning and critical thinking that she expected from students, and consciously guided discussions and problems

toward connections between topics. As described above, she emphasized how, for her own understanding, she needed all the connections and justifications to be made, and that carried over into her teaching.

Thus, changes in Jennifer's practice mirrored important themes in her own learning experience in the RLE program. She attempted to clarify her explanations and contain the focus of her lessons to avoid students feeling the frustration she experienced at the beginning of her project. She added more exploration to a small extent in the lessons planned during the interviews and to a larger extent in her geometry and algebra 2 (during the previous year) courses. Finally, she tried to emphasize connections and justifications for her students. These were characteristics of effective teaching and learning that she believed in before the program, but it seems that her experience over the summer led her to a greater emphasis on them. Furthermore, experiencing these aspects of learning as a student led to a greater empathy for her own students, which shifted her perspective regarding her conceptions of student learning. The conceptions themselves did not change, but, by the third interview, the way she discussed them focused far more on the experience of the student in the lesson.

The fact that her primary conceptions did not seem to change during the interviews is consistent with her scores on the belief survey, which were largely unchanged from the beginning of the summer to the end. Only her category I (the degree to which students construct or receive knowledge) score shifted more than 0.25 points. The change was not very significant, and it is difficult to determine the reasons for the change given the data from the interviews. Her

hypothetical learning trajectories did not show any significant shift in beliefs related to that category. However, her desire to structure explorations enough to avoid too much confusion and her adherence to a teacher-directed lesson structure while nevertheless professing a belief that students construct knowledge individually seems to indicate a mostly neutral position on category I beliefs. Her second survey response score was indeed neutral in category I, while her first was only slightly above neutral. It remains unclear whether this indicates a true shift in beliefs or not.

Finally, while Jennifer's participation in the research project and the RLE program as a whole impacted her conceptions and her practice, it is possible that the impact was mitigated by her feelings of being overwhelmed, overbusy, and distracted. She noted several times that there was a significant "time management issue" during the second summer, and that she did not feel like she had a chance to reflect upon and absorb the experience during the summer. During the first summer, she had spent a great deal of time considering how she might incorporate some of what she was learning and experiencing into her classroom, but had not felt able to do so during the second summer until the final week or two. Her abstract algebra course only compounded that issue. During the third interview, she described her school year as "just survival", and she had not had the chance to work on incorporating aspects of the RLE program into her courses as much as she would have liked. Thus, it is possible that the impact of the RLE program experience was minimized by her many other responsibilities and commitments. Nevertheless, as with other participants, her conceptions,

perspectives on her conceptions, and teaching practice were impacted in ways that mirrored her own experience as a learner.

### **Joyce**

#### **Background**

Joyce was participating in her second summer in the RLE program, and was also enrolled in a master's degree program at the university where the program was held. This program incorporated the summer RLE program into the coursework, and included a commitment to eventually return for a third summer<sup>22</sup>. At the beginning of the project, she had been teaching mathematics for two years at a large urban high school - a job she had taken immediately after finishing a bachelor's degree in mathematics. The school as a whole served a large low-income minority population with many students for whom English was a second language, and most of her students had an academic focus outside of the STEM disciplines. During my visit to her classroom, I observed that her students often needed prompting to participate in class and could be difficult to control.

Joyce taught mostly algebra (both honors and college prep) and prealgebra, courses, consisting of mostly freshman students, though she had taught an AP Calculus course the previous year. Her prealgebra classes met for a double period (periods were 45 minutes each) and she described the

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<sup>22</sup> During the third summer, she would participate in a geometry course (similar to the course taken by the third-year teachers who were present during the summer I observed the program).

curriculum as “context-based, so everything they do has a story behind it” and “slightly in the style [of the RLE summer program]”. She said that she “really like[d] it”. The second half of each of these double periods was spent in the computer lab working through a individualized computer-based curriculum designed to allow students to review and learn prealgebra topics, beginning with multi-digit multiplication and long division and continuing through the course content, at their own pace. She said it was “very well-structured to give them the support that they need, but it also allows for a lot of individuality” and commended the “instant feedback”. However, she expressed some reservations that “it’s a really good support to have now, but I don’t know how we ease out of that by the end of the year, when they’re ready.” The algebra class used a more “traditional” text that she did not like as much, and so she supplemented the lessons with additional material and incorporated “weekly investigations” that she made up on her own and consisted of “larger problems that would have a more extended thought process that went into them and more problem solving involved in them”. Before the third interview, I was able to observe two prealgebra classes.

Joyce was attracted to the RLE program as a means to earn a master’s degree. She said:

I was looking for a program that was going to give me more than what my undergraduate education had given me because I already had my initial license [...] so that was a big pull for this program that there was a lot of math involved in it but it was also an education program.



Indeed, Joyce exhibited a greater degree of mathematical sophistication than most of her fellow participants. Having recently completed an undergraduate degree in mathematics with an education minor, she noted “there were a lot of things that were very fresh in my mind that are not for other people that are here.” This was evident when she described her first summer experience:

And so we [she, her project partner and another teacher who did not return for the second summer] were able to work through a lot more problems than the average person gets through just because we remembered a lot of things, so it wasn't a recreation of all this, it was just bringing it back from rather recent memory. So we [...] spent a lot of time going through and getting as much done as we could. Which meant that we did all of the numerical problems, [...] and then we were able to crunch through a lot of the proofs, too.

As a result, she and her project partner Chad spent a lot of time during the second summer on the more intensive proofs and exploration problems. In many cases, they were the only participants to attempt certain problems.

Based on her experience in the first summer and the school-year seminars, she began incorporating something she called “‘True, False, Fix’ [where] you had to decide if some statement was true or false and if it was false you had to fix it.” This was based on a problem style used extensively in the problem sets called PODASIP (Prove Or Disprove And Salvage If Possible). She made up cards with the statements and assigned them to students “depending on what level [she] thought they were at”. She was surprised when:

The ones who did best, who were the most involved in it, were the ones who I thought were going to have the most trouble [...] they were the ones who knew to go to that concrete example first, and then try and think about it. And so that was a really great experience for me and I would like to incorporate more things like that on a regular basis.

The idea of starting with concrete examples, and then trying to make conjectures based on that evidence, was a very important one for Joyce over the course of her summer experience. Along with the importance of group and partner work for mathematical discovery, it was a dominant theme that emerged from our discussions of student learning.

### **Joyce's Summer Learning Experience**

Joyce was only two years removed from finishing an undergraduate degree in mathematics when we met at the beginning of the summer, and, partly because of this, possessed more confidence and exhibited more mathematical sophistication than most other participants. As she said, there “were a lot of things that were very fresh in my mind that are not for other people [in the program]”. During the first summer, she worked closely with two other new teachers who had also recently completed bachelor’s degrees in mathematics. One of these teachers did not return for the second summer, but the other, Chad, did, and the two of them worked extremely closely together. They were paired together for the research project and also worked together on the problem sets. Though they sometimes discussed problems with others, they always worked together, often on problems that other participants did not feel prepared to tackle. When asked to discuss what it was like to work in pairs on the research project, Joyce found it difficult to imagine working without Chad, saying that “obviously [we] work very well together [...] but working with him is just, like, what I do all the time, so I don’t really know what to say about that!” When she discussed their results and the process of arriving at them, she repeatedly used the pronoun

“we”, indicating that she viewed their discoveries and development as a joint enterprise. When asked what it would have been like to work with a different partner, she noted the value of their partnership:

Overhearing what some of the other groups were doing, it sounded like they kind of went their separate ways and did work and kind of came back and discussed it rather than working together all the time. In some ways it might have been helpful to have the space to go in different directions, which we did sometimes. Like, we’d be sitting next to each other and going in different directions, but I think it’s really nice that we were working together all the time because even if we were doing something but I had an idea I could bounce it off of him at any time [...] It’s nice to be able to just think of something and before I forget it, say it!

She quickly refuted her sole critique of their partnership, that “it might have been helpful to have the space to go in different directions” by noting that they did this, just in close proximity. She valued the opportunity to bounce ideas off of Chad and to similarly provide a sounding board for him. They also periodically consulted the program counselors, and she said, “it was nice to have different perspectives [...] they brought in a lot of different angles, a lot of them which we weren’t really willing to go into.” Though they did not rely on outside input as much as they relied on each other’s, she saw a clear value in consulting outside sources, and even noted that at least one counselor’s input had provided a direction that had proven useful to them.

Joyce continued her master’s program in the fall and was required to complete a research project alone. During the second interview, she was worried about this, saying she thought she would “be calling [Chad] on a regular basis and being like ‘hey, so I’m trying this and this happened and what do you think?’” She noted that she liked “to talk through what I’m doing”, which made

working alone “challenging”. When we met for the third interview, she had started her master’s project and confessed that working alone caused to feel “not very motivated.” She worked with a mentor professor, but their communication was mostly limited to e-mail, which made it difficult to even describe her results. Though the project continued after our final interview, at the time of our third interview, Joyce was not nearly as enthusiastic about it as she was about her summer project, partly because working alone made the work less compelling and rewarding.

The importance of her partnership with Chad illuminated Joyce’s emphasis on the role of group processes in student learning, an emphasis that grew over the course of the summer. She considered herself and Chad to have developed their results jointly, and noted the way their discussions and interactions contributed to their mutual knowledge growth. Joyce consistently valued group work in her classroom, and her perspective on its value shifted over the course of our interviews. The sections below detail the way this occurred.

Similarly, she began to emphasize exploration, calculation, and data collection as a key part of student learning and mathematics research, and this had its roots in her summer research experience. She admitted that she “was expecting [the research project] to be much more structured than it was.” She was surprised by the open-ended nature of the task, and she and Chad initially struggled with the lack of structure. As she put it, all they were given was “here are these functions, why don’t you look at them. Maybe you might want to look at them in mod  $p$ . Go for it!” As a result, they started by “just churning out all this

data,” doing “a lot of examples - a lot, a lot of examples”. After doing this for some time, they started to feel frustrated. She described their frustration by recalling their feelings at the time: “we don’t see anything! We’re not really sure where we’re going! This project is not going to work out!” However, reflecting on that experience gave her insight into the nature of mathematics research:

It gave you that sense of, like, research is not handed to you on a platter. When you do research, you have to find a lot of information and then sit down and organize it and then realize that you don’t have enough yet and go off in a different direction and ask different people for help.

For Joyce, the initial work of mathematics consists of familiarization, experimentation, and reflection. Others play an important role in directing and aiding in that effort. However, the route from the initial experimentation to the eventual conclusion was less clear for her:

But then suddenly, magically, you know, things just started falling into place and once we had one thing fall into place, we had one conjecture that we were able to prove, we were like “oh, well then, based on this, we can do all these other things!”

This “magical” moment was exciting, but somewhat mysterious to Joyce. She seemed to attribute it to the experimental work they had been doing, but wasn’t clear about what triggered it. Still, it was clear to her that the numerical experimentation and pattern-sniffing (Cuoco, Goldenberg, & Mark, 1996) instigated the “ah-hah” moment. She appeared to attribute the learning to the activity without understanding how the activity instigated cognition that led to learning. For her, the mechanism, the means, of learning, rather than in-the-head work, was paramount.

In the third interview, Joyce summed up what she learned from her research project as follows:

I learned math can be a lot of work. And really the struggle. Because for most of my math career, math has not been a struggle. And so that perspective of [...] it takes a lot of thought and figuring things out and finding patterns. And just because things are really obvious to me about algebra foundations stuff doesn't mean it's going to come immediately to these kids.

The struggle of exploration in order to find patterns was clearly meaningful to her, and she drew parallels between that experience and her students' learning. That is, she projected her own experience onto her students in order to understand how they react to unfamiliar content. The key role of experimentation in facilitating and inspiring learning would also come to play an important role in her conceptions of student learning as the result of her own experience of gaining insight that she attributed to experimentation and data collection. Below, I will detail how Joyce's summer experience led to changes in her conceptions of student learning. In particular, the fact that social interactions and experimentation played important roles in her own learning led her to emphasize these ideas for her own students' learning.

### **Belief Inventory**

Joyce's responses on the belief inventory indicated a high level of agreement in all three categories: with the notions that students construct knowledge for themselves (category I), that research mathematics and student learning in mathematics are similar (category II), and that students are capable of significant independent insights (category III). Her response scores in each category remained largely unchanged between the two survey administrations:

TABLE 4: *Joyce's Belief Survey Response Scores*

	Adjusted Category I	Adjusted Category II	Adjusted Category III
Pre	4.222	4.0	4.5
Post	4.222	4.0	4.25

These scores are consistent with the beliefs she professed during the interviews, and the lack of significant shift in any one category indicates that her beliefs about student learning remained largely unchanged. The organization and principles of the RLE program, as well as her experience with the program, were consistent with her preexisting belief systems. Thus, her time in the RLE program reinforced rather than challenged her beliefs that students construct their own knowledge, are independently motivated and capable, so they did not change significantly. Nevertheless, some of the mechanisms she identified by which student learning proceeded changed in ways that mirrored her summer experience. Next, I will describe Joyce's beliefs about student learning, and the ways in which her emphases changed, by discussing her lesson structures and hypothetical learning trajectories over the three interviews.

### **Hypothetical Learning Trajectories**

Joyce's first lesson was on solving linear equations. She looked at the sample lesson, but set it aside and chose to construct her own, using her school's prealgebra textbook as an inspiration. Though she did not explicitly state goals at the outset of her lessons, the implicit goal of the first lesson was to model a situation using a linear equation and then to use that model to gain more information about the situation and it remained consistent throughout all three interviews. For her lesson on defining *function*, the goal seemed to be to help

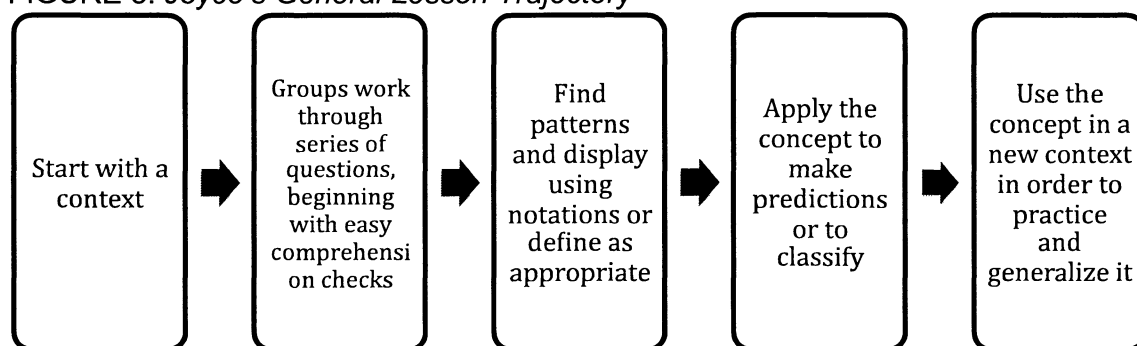
students understand the concept as a correspondence between two sets, where each element of the first set is assigned to exactly one member of the second set, and to use this definition to determine if a given correspondence was a function. This, too, was consistent in each of the three interviews, and was modeled after her textbook, though she did not recall the textbook lesson as precisely for this particular topic.

Joyce's deference to her textbook is indicative of her agreement with its philosophy. She seemed to take the goals of the lesson as given, and did not question or attempt to extend them. The basic trajectory of both lessons began with a "context paragraph" to be read together as a class. The lesson on solving linear equations used a story involving plant height increasing over a number of days, and she used this as the motivating context for the lesson on defining *function*, as well. Next, students worked in groups through a series of questions that required increasingly sophisticated manipulation of the data given in the story. These questions began with something very straightforward ("nice, easy question"), such as "how tall is the plant on day 7?" – a straightforward reading or extrapolation of the chart. The questions eventually moved toward prompting students to "look for the pattern [...] see if they can find that pattern", such as "find an equation that will give the height of the plant on any given day." Eventually, this sequence was supposed to lead students to applying the concept (a linear equation in the first lesson, functions in the second) to make predictions in the opening context, such as predicting how tall the plant would be after a large number of days, interpreting the slope and y-intercept of the equation in



terms of the plant's growth, or determining the day on which the plant would reach a certain height (solving the equation). In order to practice and reinforce the concept, students were then asked to apply it to new contexts. For instance, a new story might be provided for homework, with some of the same types of questions asked in the new context as in the old. Thus, Joyce's lesson plans during the first interview exhibited the basic structure shown in Figure 3:

FIGURE 3: *Joyce's General Lesson Trajectory*



During the second interview, she added a preliminary step to the lesson on functions – “sort of [a] motivational something of why we care if something is a function [...] something so that we know why we would ever want to define such things”. This was essentially an extra context paragraph, using the relationship between days of the year and recorded high temperature to motivate the idea of a function. The lessons I observed during my classroom visit followed a similar trajectory, unsurprising given the close adherence of her lessons to her textbook.

Thus, the goals and substance of the lessons did not change significantly over the course of the program. Nor did the learning trajectories she identified for students participating in the lesson. In all cases, she indicated that a typical student would essentially progress linearly through the lesson she described.

That is, students would familiarize themselves with the context and develop a sense of the motivation behind the concept. They would work through the series of questions, incrementally building their understanding in a logical series of steps, first by answering straightforward computational or numerical questions, and then generalizing, developing “intuition”, along the way. Once they were able to do that, they would repeat the sequence in a new context in order to practice and demonstrate mastery of the concept. She expected that students were sufficiently motivated and capable of achieving this. For example, in interview one, she illustrated how a student develops knowledge over the course of a lesson by describing a student solving a linear equation that relates the number of days since acquiring a plant to the height in centimeters:

Because they wrote the expression themselves, it should be pretty intuitive for them to know how to use the expression, whereas if I'd given them an equation, they would have to figure out what the equation really meant and what that variable was and where it was coming from, [...] Then they should have the ability to say “oh, well, centimeters was the answer that we got.” So at least they have some sort of connection between that they wrote this expression and then they got an answer, and will in some way be able to work backwards [...] So [the student will] use what they know because they just figured something out, they know how to go from the number of days to the number of centimeters. So if I give you the number of centimeters, use what you just figured out, and so, I don't want them to be totally reinventing the wheel all the time, I want them to be sitting back and being like ‘oh, well, I've done something like this before, so let me see if I can use that.’

Thus, her lesson structure was based on a prediction of how students would put together the concept. The lesson structure itself provides a trajectory for student learning. Discussing the same part of the lesson during the third interview, she

identified a similar process. Note the similar reference to intuition and how it is built and then called upon to advance to the next step in the lesson:

And then, taking that intuition, previous experience, and way that they've solved things before and then applying that to writing an equation. So then they can see how that relates to the new method. And so, by having some intuition before they're given the new experience, then they can really see that "oh, the answer that I get when I do it this way makes sense because that's the answer I would have gotten when I did it the other way, also"

Joyce's hypothetical learning trajectories were significantly learner-driven in that they were designed around student learning trajectories for specific content. However, her expectation was that student learning would not deviate from this trajectory, and that knowledge is constructed as a result of an incremental, step-by-step progression through increasingly sophisticated problem-solving. The development of "intuition" for working with mathematics was key in this process and the instrument by which learning progressed, though she believed intuition developed by virtue of certain learning activities and was unclear how exactly these activities led to that development. However, though the hypothetical learning trajectories she constructed did not change over the course of the three interviews, the mechanisms she identified by which students progress through the trajectory did. In particular, she began to place more emphasis on the importance of social interactions for knowledge development, and on the development of intuitions through data collection, calculations, and exploration. Both were key aspects of her own learning experience during the RLE program. The latter mechanism played a significant role in her conception of mathematics research, as well.

## **Mechanisms By Which Students Learn**

### **Social Interaction**

As described above, Joyce expected students to often call on their mathematical “intuition”, or “to use what they know” in order to solve unfamiliar problems and thereby advance their mathematical knowledge. Other mechanisms for proceeding through the trajectories described above shifted over the course of the three interviews.

Joyce’s classroom utilized a great deal of group work. When students were working through the problem sequences, this was always done in groups: “we spend a lot of time in this style of, like, you work in groups, you talk to your partners, you figure things out and you work things through.” At the outset of the program, however, her focus was on the individual actor in that group. That is, she believed group work was important, but indicated that knowledge construction was essentially an individual process:

Most of those connections are going to be made when they’re actually doing the work [...] If somebody else is doing something, you can nod along and say that makes sense, but actually creating things for yourself is very different than following somebody else’s change of logic because you have to be able to actually create that logic yourself. (interview 1)

The group, on the other hand, principally functioned as a resource for the individual rather than as an integrated knowledge-constructing agent:

Since there’s only one teacher and there’s twenty kids, it’s a lot easier if they have a group to turn to than if they need to ask the teacher every time they have a simple question. And so I’m bouncing around the room making sure that their friends are defining [the term] correctly, but it’s nice that they’re able to look to each other and see each other as resources. And also that [...] it’s

very useful for them to learn from each other and to learn that there's not always just one way to solve the problem. (interview 1)

Thus, group work was not viewed as the collective discovery or creation of knowledge, but instead as an organizational construct that facilitated and aided individual knowledge construction.

At the end of the summer, the second interview indicated that she still held this view of group work. However, the focus was subtly different. Her discussions focused more on the social interactions that would take place within the group. For instance:

I let the kids talk all the time and we spend a lot of time in this style of, like, you work in groups, you talk to your partners, you figure things out and you work things through. And when I approach kids, I'm not approaching them with pencil and paper and showing them what to do, I'm approaching them with questions and asking them to have discourse and to discuss things [...] there's definitely a very talkative, like, community aspect of how things work in my classroom [...] talking is something I really depend on. (interview 2)

While individual knowledge construction was still paramount, the context in which it took place was characterized as more social than before. Rather than the group simply acting as a resource to the individual student, social interactions were seen to play a more active role in the individual's learning. She was describing her previous work in both interviews, so this change doesn't indicate a change in teaching philosophy or even, necessarily, in her beliefs about how students learn. However, it does perhaps indicate her perspective on the role of the group shifted over the course of the summer. This was illustrated when I asked Joyce to verbalize the mental processes of a hypothetical student

participating in the lesson on linear equations. She referred to the hypothetical student as “Mike”:

He'll [Mike] be able to use that expression, but he might be a little stuck and he might try counting backwards even numbers for a while and lose count and think there might be another way. And maybe ask his partner and see what his partner's thinking and figure out that, “oh, well, this equation that they use up here gave the eight, so maybe you could use that equation somehow” [...] And so Johnny who's sitting next to him is going to have some inkling of what to do. And Mike's going to figure it out and he's [Johnny] going to help him and then he's [Mike] going to figure out how to work backwards. So they're going to work backwards using that equation. (interview 2)

Even when discussing an individual learner, “Mike”, group processes and interactions with “Johnny” played a key role in learning. Rather than Johnny being a resource to Mike, they together make strides toward a new method for solving linear equations – “working backwards”. While learning was still individualized, it was seen as a collaborative effort.

While her perspective on group work underwent a subtle shift, Joyce's view of learning as a principally individual process was consistent over the course of all three interviews. However, the emphasis on the important role of peer interactions in individual knowledge construction remained intact. When discussing the computer program her prealgebra students used during part of their class time, she professed support for the program's “student-centered” approach and “instant feedback”, saying that it's “actually a really awesome thing because it totally individualizes everybody's mathematical experience”. But when I asked if she thought it would work as a replacement for all class work, she disagreed very strongly:

No, that wouldn't be a good idea. I don't think that would work at all because they don't get any of the peer feedback. Very few of them work together when they're on that [program] [...] there's just a classroom dynamic that's really missing when you do everything like that. (interview 3)

While she was supportive of the computer program's individualized nature, she considered it lacking because it did not allow for peer feedback – something she considered vital to student learning. She considered social interaction to be key to student learning, and this belief was strengthened over the course of her summer experience. As discussed earlier, peer interactions played an important role in her own summer work, and may have played a role in that shift.

#### Developing Intuition Through Experimentation

In addition to group interactions, Joyce repeatedly emphasized the importance of “developing intuition” about mathematics, identifying it as an instructional goal. She used the term two ways. First, to refer to students' prior conceptions and existing knowledge: For example, when outlining one lesson, she stated, “[at the beginning of the lesson,] I want to figure out what their intuitions are and what their initial assumption is, so that way if there's anything that they do incorrectly, I can fix it.” Also, when discussing the definition of *function*, she mentioned the importance of acknowledging existing preconceptions of terms: “there's going to be a lot of language in there that is in your [the student's] intuition, but needs to be stated explicitly here.” She also used the term to refer to their developing mathematical understandings. For example: “I had a curriculum that was set up so the kids could read it themselves and get information and build intuitions and things like that.” Intuition was thus

identified as the set of existing conceptions that shaped a student's participation in a lesson as well as those developing conceptions that were a product of that participation.

In that first interview at the beginning of the summer, references to the latter were limited to its development as a byproduct of the lesson without any attention paid to the mechanisms by which it developed. She spoke of waiting to introduce the vertical line test "after they've developed some intuition" and identified the opening activity of the linear equations lesson as an opportunity to "develop that intuition of how these things work" without discussing how that development occurs.

In the second interview, Joyce reaffirmed the importance of intuition, saying, for instance, she wanted "to make sure that they have a lot of intuition of what that [the idea of a function] is before we give the definition [of function]." She also emphasized that it was important to provide students opportunities to explore independently so that they can develop personal intuitions about the concept:

I would really try to do the minimum number of examples possible as a class, because the kids who have figured it out [...] are just going to be shouting out answers and giving things away, and that doesn't really let the other kids explore and develop the intuition and understand things.

Note that exploring was linked to "develop[ing] intuition and understand[ing] things". She linked the development of understanding to the opportunity to explore. When asked if she'd thought about her students while doing her project,



she again emphasized that exploring multiple examples was key in order to generalize, understand, and gain insight:

Doing that sort of like generating a lot of data is useful not just when you're doing a big project but also generating data is a good idea when you're doing just general things [...] Developing data, but also knowing what data is important to develop and then making generalizations. Even if we're making generalizations that they've probably made before in the eighth grade, that everyone knows. But they're still generalizations that are important to figure out on your own because then they have a better understanding of them.

Though she did not use the term explicitly in this quote, Joyce described the initial work that went into developing intuition – collecting data and making generalizations that should prove useful for problem solving.

Joyce also advised her students in creating projects for a school math fair, and her new emphasis on developing intuition through exploration and data collection was apparent in our discussions of these projects. The mini research projects student developed and carried out had been a big part of her work the previous two school years. We discussed it during the first interview, and she was obviously proud of her students' work and excited by the learning these experiences evoked. At the end of the summer, however, she noted that during the upcoming school year she "intend[ed] to do it differently." In particular, she noted that she liked "the idea of just having them do a lot of examples [...] it] would be a really effective way for them to spend their time, to develop that intuition and get some insights." Later, she added that "maybe even before we do hypotheses, we'll just be getting a lot of information." This was a shift from the previous model she had used, and was consistent with a new emphasis on exploring and data-gathering as a means of intuition development.

During the third interview, Joyce retained her emphasis on intuition as both the starting point for new learning and the outcome of mathematical exploration. She discussed how students move through a lesson by first applying existing intuitions, then developing new ones that, in turn, are used to solve more sophisticated problems:

So by using their previous experience and whatever their intuition tells them, they can figure out how they really want to be solving this [linear equation]. And then, taking that intuition, previous experience, and way that they've solved things before and then applying that to writing an equation, so then they can see how that relates to the new method. And so, by having some intuition before they're given the new experience, then they can really see that 'oh, the answer that I get when I do it this way makes sense because that's the answer I would have gotten when I did it the other way, also.'

On the day of observation and the third interview, the lesson was an introduction to adding fractions. They began with several simple common-denominator fraction addition problems that were supposed to lead students to generalize a rule for adding fractions with common denominators. I did not see the completion of the lesson (it extended until the next class meeting), and the students in the class were not fully engaged in the lesson, but the structure adhered to the idea of exploring in order to develop intuition about the topic.

Joyce utilized the term "intuition" to call to mind both the sum total of prior experience and learning and the understanding gained. The mechanisms she identified by which it develops were outlined more clearly at the end of the summer session than they were at the beginning, with an emphasis on exploration, experimentation, and data-collection as means for students to develop intuition about a particular subject. Her own learning experience in the

RLE program drew her attention to the role these actions can play in learning, and that was reflected in her discussions of how intuition develops. Just as her shift in emphasis regarding the role of group work in learning was rooted in her own participation in the research project, so too was the change in the way she talked about exploration and experimentation. Below, I will discuss Joyce's conceptions of mathematics research and its relationship to student learning. These conceptions developed over the course of her research project, and many of the elements of that experience that she drew upon when considering student learning also shaped her notions of mathematics research.

### **Joyce's Conceptions of Mathematics Research**

During the second interview, Joyce described mathematics research as "exploring an idea and finding patterns and relationships." She was asked to respond to a hypothetical situation wherein a student asked her what mathematics research is. Her response illuminated her beliefs about the nature of mathematical creation:

[Y]ou're looking at something and so you pick a topic and then study it a lot and figure a lot of stuff out, which probably involves doing a lot of problems and calculations and stuff like that. And then trying to see what it means in some underlying [structure] within it.

She emphasized the initial familiarization work that Hadamard (1945)<sup>23</sup> and Ervynck (1991) identified as the first stage of mathematical creation. She was less clear on the nature of the ensuing generalization, but again stressed the

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<sup>23</sup> Hadamard was echoing the work of Poincaré

preliminary data collection and calculation phase. When asked how she would have answered differently before the summer program, she responded by saying “there would have been less emphasis on doing calculations and more of an emphasis on seeing generalizations.” Her summer experience, as outlined above, shifted this conception, leading her to conclude that “the research part is generating the data and knowing what data to generate, and then finding patterns.”

Furthermore, Joyce never considered the mathematics research to be limited to the realm of professionals and academics. In her initial lesson plan on solving linear equations, Joyce had students modeling the growth of a plant using a linear equation, and wanted to encourage students to represent the quantities using variables by telling the class that “mathematicians use variables as shorthand.” She said, “in that case, I’m including myself as a mathematician and the students as mathematicians, and I want them thinking they get to be mathematicians, too, so they can do the same things that mathematicians do.”

During the second interview, she expounded on this, saying that she

want[s] to have them think about, like – they [the students] are people who can do math, therefore they are mathematicians and that that big, fancy title is not reserved for people who have PhDs. That anyone who does math can be a mathematician and that they are doing substantial math.

Her belief that students could be counted as mathematicians and could behave like mathematicians was consistent with her beliefs about the relationships between mathematics research and student learning. As described above, she characterized mathematics research as a process consisting of exploration and

data collection in order to develop intuitions that lead to generalizations. By including students as mathematicians, she indicated a belief that student learning progressed along the same path. In fact, in the third interview, when asked to respond to the statement *when mathematicians do mathematics, they're doing something fundamentally different than when students do mathematics*, she disagreed, saying:

I think there's the same process of exploration and constructing ideas. I think that when students do mathematics, it's much more structured than when mathematicians do mathematics, because the teachers are directing them in a particular direction and sending them to a particular purpose. But I do think that there's still that same idea of you start with a problem and you don't know what the answer is, and you've got to figure it out. So in many ways, what the students are faced with is just as foreign as a mathematician who's facing a problem that nobody's ever solved before. Because they've never solved it before, so it's something new to them.

Though she acknowledged that the teacher exerts much more control over the process (a point made by Ernest, 1998b) and that students were not making discoveries new to the field as a whole, Joyce nevertheless characterized student learning and mathematics research as parallel processes. She went on to claim that it was possible that this was not true in every classroom, but was in her own classroom, which was structured to “find patterns and tell me what's going on and make generalizations, which is what you're doing when you're doing research.”

Thus, Joyce saw mathematics research as the process of exploring and collecting data in order to develop intuitions, and making generalizations from that. She noted that this attention to data collection and calculations developed over the summer, and believed that students learning high school mathematics

were engaged in essentially the same process as research mathematicians. Her experience with the research project influenced these beliefs, in part because her own learning followed her perceived trajectory of mathematics research. As she drew parallels between her own learning and that of her students, she also built upon her own experience with research in order to determine the nature of mathematics research. Thus, parallels developed between her conceptions of the processes of mathematics research and student learning.

### **Discussion and Conclusion**

Joyce's primary belief that students learned through individual construction of knowledge, which develops through problem solving and experimentation, stayed consistent over the course of the summer program, as did the her general hypothetical learning trajectory for her students and her emphasis on the role of intuition as both a prompter and product of learning. Over the course of the summer program, some of her beliefs were reinforced, and her conceptions of the mechanisms of knowledge creation changed in response to aspects of her experience that were particularly meaningful. In particular, her belief in the value of group work was reinforced, while her perspective on the role of those groups in knowledge construction shifted from the idea that groups were simply an outside resource that the individual could periodically consult to the notion that the peer interactions play a crucial role knowledge development. Furthermore, she began to emphasize initial experimentation and data collection as the first phase of mathematical discovery,

echoing Hadamard (1945). This was characterized as a vital part of both mathematics research and student learning, processes that she saw as similar.

The roots of these shifts can be seen in her own mathematics research experience, where she worked extremely closely with a partner and spent a great deal of time on calculation-based exploratory work. It was her personal experience that caused her to rethink or flesh out her conceptions of student learning. Furthermore, her experience reinforced and further developed her belief that mathematics research and student learning are similar processes. Joyce's belief survey responses indicate that she held this to be true even before the research program. However, her own research experience led her to stress those aspects that were meaningful in her experience – in particular, the idea that research and learning begins with data collection, eventually leading to knowledge construction. I argue that the parallels she saw were the result of her processing her own summer experience. Her conception of mathematics research was very much limited to her summer project. She emphasized the role of exhaustive calculations for familiarization (see Hadamard, 1945; Ervynck, 1991), individual intuition (see Thurston, 1994), and social interaction (see Boaler, 2002) – all important aspects of the mathematics research process. However, she minimized the important reasoning processes involved (see Dreyfus, 1991; Cuoco, Goldenberg, & Mark, 1996), as well as the steps of proving and disseminating. Her discussion of mathematics research was limited to those aspects that were of particular importance to her own experience. Thus, the parallels she drew between mathematics research and student learning came

about as a result of her own learning in a context described as mathematics research. She saw parallels between her own experience and that of her students, and therefore appropriated the context of her experience, mathematics research, as similarly parallel. Further supporting this claim is the fact that changes in her perspective on group work coincided with a learning experience where social interactions played an extremely large and significant role.

Furthermore, her continual appeal to “intuition” indicates that she was unsure how exploration led to the “ah-hah” moment – she felt it was important, but the origins of that moment were somewhat mysterious to her. She was unable to describe how it would occur for her students because she was unsure how it came about for herself. She was sure, however, that exploration and “data-collection” led to the “ah-hah” moment, so she attributed this as a property of the activity (Heinz, et al., 2000).

Thus, Joyce’s primary beliefs about student learning were not impacted by her experience in the RLE program. However, her beliefs regarding the mechanisms by which learning proceeds did shift along with her conception about mathematics research. As she learned through a research-like setting, those things that prompted her own learning emerged as important aspects of how she believed both processes – research and student learning – proceeded. The changes were the result of her drawing upon her own experience. At the same time, since that experience was “research-like”, the notion that mathematics research and student learning were similar was reinforced for her.



Her conceptions about the ways students learn were changed by exposure to research through the filter of her own learning experience.

## **Emily**

### **Background**

Emily was in her first year in the program and provided some insight into the experience of these teachers. Though she (like other first-summer teachers) did not participate in the research project, her experience in the number theory course contained a number of elements of mathematics research. In particular, the problem sets were designed to encourage “thinking like a mathematician” by encouraging generalizations and conjectures based on computational experience and by motivating the need to axiomatize - reduce assumptions to a minimum number of axioms. Furthermore, organizers stressed the idea of behaving like a mathematician by “experimenting” with mathematical objects just as a scientist experiments. Thus, even first-summer teachers were exposed to some aspects of mathematics research, though the projects were more structured and not as long-term as the projects completed by those teachers in the second summer.

Prior to starting the program, Emily had six years of teaching experience at a private religious high school, and she began her first year in a new position at a public suburban high school during the school year immediately following her summer in the RLE program. The observation and third interview were conducted at that high school. Her undergraduate major was mathematics, which she believed to be an asset during the number theory course, and she had initially intended to “do something computer science oriented” after graduating.

Some summer internships led her to conclude that that work was not “stimulating or rewarding at all”, and she decided to go into teaching a year after finishing her undergraduate program. She participated in a certification program focused on elementary education. In her words: “I knew I didn’t want to do that [teach elementary school], but I just had so little teaching experience, like none, that I wanted something that was a little more structured.” She then “jumped into the private school world and started teaching [...] they just let me do what I wanted.” She claimed that for the first two years, she “had no clue as to scope of what I was doing or where it was going, or pacing” and, over time, “kind of figured stuff out on my own [...] kind of just picked it up along the way.” These first years of teaching, along with her time as an undergraduate mathematics student, were extremely influential in determining her teaching philosophy and her conceptions of student learning

After three years of teaching, she enrolled in evening classes toward a master’s degree in mathematics education, which she had recently completed prior to the summer we met. After moving, she had spent the previous school year working part time in a middle school setting. Emily seemed to feel that the majority of her teaching philosophy and teaching knowledge had developed over the course of her first years of practice. She cited her lack of experience prior to teaching and the struggle of the first few years, coupled with some teacher mentors, as the key aspects of her development as a teacher. She thus valued learning through her own experience, and had largely drawn on her own

experience as a learner in those formative years of practice when she was mostly left to her own devices.

Prior to the RLE program, Emily described her classroom as conversation-based, but teacher-centered. That is, she stood at the board and guided a discussion with her students:

The time I spent with them was very much about a conversation and a dynamic kind of like, "I'll start you off with a question and then I want to know what you think about that". I definitely think like the visual aspect and the board is a key part of the classroom, but I wouldn't say that it was me being frontal and speaking. It was just, like, that's where the board is [...] So there was a lot of students giving suggestions, students commenting on other people's methods, a lot of questions."

Her classroom was very teacher-directed, with all student input directed to the front of the room and responded to by Emily. Student feedback was encouraged, but Emily was the focus of the classroom and directed discussions. She characterized her classroom as a place where "there was a lot of student giving suggestions, students commenting on other people's methods, a lot of questions," where she would always require students to justify and explain their reasoning. This was true during my observation, though there was not a great deal of give-and-take between students. Responses and questions were directed at Emily herself. As I shall discuss below, her lessons, both those developed during the interviews and those observed in the classroom, were principally focused on developing and mastering procedures, and reflected her own learning experiences and conceptions of mathematics.

Emily found out about the RLE program when she stumbled upon it online, and she saw it as a good opportunity to learn more mathematics. Later, she

would comment that she came to the program to “learn something and get a new perspective on learning and teaching math.” She characterized the first few weeks of the program as “a positive experience” and, after a week of working on number theory, described her initial reaction as follows:

During the time I was working with seventh and eighth graders, I didn't find it to be as intellectually stimulating, so now I feel like my brain's getting back – it's like I'm working out again. So that's something I'm really, really appreciating is just that I'm put in a position that I haven't been in a while to really exercise my own understanding of mathematics which is a little bit out of reach for me. So it's like struggle for me, which I appreciate because I haven't felt that for a really long time.

She was clearly feeling challenged by the work and appreciated the opportunity to participate as a learner. Her summer experience is discussed in detail below.

### **Emily's Summer Learning Experience**

Emily was drawn to the RLE program because of the opportunity to learn new content and to experience a new perspective on teaching, and initially found it stimulating to “exercise her brain”. At the end of the summer, she still felt that learning had been valuable and, while she found it “difficult” to encounter “a lot of things that I'd never studied at all before or some things that I vaguely remembered but it didn't really come back to me very easily,” and admitted “struggling”, she saw value in the experience. In particular, she cited the opportunity to experience learning new material as a student as an “interesting” aspect of her experience:

It's an interesting perspective to be back in a situation where you're really struggling with something and you don't understand it. The things that I've been teaching I understand very well, even when I was first teaching them and I wasn't completely inside it and I didn't know how to explain it, I still felt like I could have done the

problems. Math itself wasn't beyond me, and this [summer content] I feel like is slightly out of my reach. I really need to work hard at putting it all together, so it's been an interesting thing for me to stop and think about, you know, how my students must be feeling if it's brand new to them, or if it seems really foreign, like "where is this all coming from, how do you put it together?"

She encountered unfamiliar content, and appreciated what her students must be feeling in similar circumstances. She noted that it was often "really intimidating, really frustrating" to be in such a situation, and that she'd "had moments where I was just totally stuck and I didn't know where to go [...] that can be a very frustrating experience, where you feel like you just have no clue, you're totally unprepared." She said she often felt at a loss but could frequently work through it. "A few times", however, she felt "completely at a loss." In these instances, she did not have any idea how to proceed, and cited the help of the counselors as key to helping her move forward. They would provide hints and she "could just *do* what they were telling me and there were some procedural things that I could work my way through, but still feeling so unsure of the process of what was going on behind it."

Note that when Emily felt at a loss for how to proceed, she turned to the "experts" available to her and followed the procedures they suggested. Simply doing something procedural helped her gain some understanding of the problem. So, in her own learning experience, applying and practicing procedures allowed her to "come to an understanding of it." She cited one particular problem that was "overwhelming." She consulted with counselors, who helped her see directions in which she might proceed, and found that

just doing more of them got me more to the end, so I came to appreciate, I mean, in the beginning, you're really just [...] punching out numbers, writing things down, you don't really know why. And then you get to be more fluent, you get to be more, you know, see certain patterns [...] And as you see more and more and more examples it sort of becomes clear.

So developing procedural fluency and practicing with it was important for her to feel as though she had mastered the material. Furthermore, she defined mastery in terms of her ability to do problems, saying that "there are things where I feel like I've totally mastered it – I could do a problem." She judged her own learning according to the same criteria that she used for to define and discuss student learning.

When asked what she felt she learned during the summer, Emily responded by saying she had learned about "experimenting in mathematics," and noted that even though she had some familiarity with "strategies and structures that you look for that are familiar", her knowledge of strategies to draw upon "expanded quite a bit." Her lesson structure, described in detail below, also emphasized building upon familiar procedures as a starting point for learning, so her own learning experience mirrored that which she expected of her students. Indeed, she said that "outside of content," one of the major things she had learned was to ask, "what can you reach back to that you've seen before."

Not every part of her experience was consistent with her classroom structure. She said, "it was definitely a new experience for me, especially lately [near the end of the summer], to struggle so much, so that was hard for me in the beginning, definitely." Her lessons were structured so that the teacher supported and initiated student learning and thus minimize the struggle that they

experienced. She was unsure if it was appropriate to allow students to experiment, explore, and struggle in the way she was encouraged to over the summer:

And also [I learned] this idea about exploration and experience before [...] revealing what the answer is or what the statement is, on a much more exaggerated timeline than I have ever done. Even if I had my students experiment with something and try it out, I would have never had them struggle through something for as many days as we did, which was a valuable experience in a lot of ways even though it was frustrating.

Thus, at least one aspect of her summer experience was inconsistent with her teaching style, and she seemed to resolve this conflict by determining that such a structure was inappropriate for her own students. Indeed, speaking about the structure of her lesson, she said, “this is not really full-on exploration, it’s like ‘you can think about it a little bit, but I’m really guiding the way.’ So I don’t know how I would free myself from that, but that’s sort of my style.” This would seem to indicate that her own experience illuminated other possibilities for teaching and learning, but she remained unsure how to incorporate these into her teaching. Her own “style” was familiar and yielded the results she expected.

Emily began work at a new school just a few weeks after finishing the summer program, and when we met again in October, she admitted that she had thought about her summer experience “hardly at all.” The combination of a new job and personal circumstances had commanded her attention and kept her from consciously processing the summer experience and, ultimately, prevented her from returning for the second summer. However, when I asked what, if anything, she had learned over the summer, she responded by saying:

I learned a lot about ... perseverance (laughs). Spending time with something that doesn't seem clear and just, like, letting patterns emerge as I work through it, or experiencing it [...] And also seeing things from a few different perspectives and then eventually having it tie together. [...] I just remember there were so many problems that I worked on that seemed totally separate [...] like, it seemed like it was totally disjoint, not related, and then eventually something, some thread was tied between two different types of problems, two different ideas that made some type of connection, and I definitely experienced that over the summer.

Thus, she noted that, over the summer, she had learned by doing multiple problems and examples and allowing patterns and generalizations to emerge.

Many themes of Emily's summer learning experience were echoed in her conceptions of student learning and the hypothetical learning trajectories she constructed. In particular, the importance of drawing on existing knowledge, the importance of following procedural directions from experts when struggling, and procedural fluency as the key criteria for mastery were all vital aspects of both her own learning experience and that which she envisioned for her students. However, as a learner, she was allowed to struggle over multiple examples without a clear expectation of the patterns that might emerge. This was very different from the structure of her own lessons, and contradicted her beliefs about the level of support that was necessary and appropriate for students. This contradiction was mostly dismissed on the grounds that it would not be appropriate to allow students to struggle to that extent, but it nevertheless had a subtle but interesting impact on her practice and her conceptions of student learning, which I now turn to discussing.



### **Belief Inventory**

Emily's responses on the initial belief survey indicated that she believed students received knowledge to a greater degree than they constructed it for themselves (category I). Out of all respondents, she was the only one with a category I score below 3. That is, every other respondent began the program either neutral on the degree to which students construct or receive knowledge or in agreement with the notion that students construct it for themselves. Her unique (in this sample, anyway) beliefs in this category were a major reason I chose to explore her conceptions in a more in-depth way through the interviews. In category II, her responses indicated agreement with the notion that mathematics research and student learning are similar processes. In category III, her responses indicated that she believed students to be capable of significant insights on their own, though her agreement with that notion was only mild. Table 5 shows her response scores in each category for both surveys.

TABLE 5: *Emily's Belief Survey Response Scores*

	Adjusted Category I	Adjusted Category II	Adjusted Category III
Pre	2.778	3.75	3.25
Post	3.0	3.875	3.125

As you can see, by the end of the program, her category III beliefs, regarding student capability, had dropped slightly, but had not changed significantly. Her category I and II scores both increased, but again, not significantly. The next section will show that the hypothetical learning trajectories Emily constructed did not change significantly, so this lack of change was consistent with other measures. However, her experience in the RLE program contradicted some of

her preconceptions about student learning, and some subtle changes in the way she discussed her hypothetical learning trajectories indicated that the summer program did have some impact on her conceptions. Next, I will discuss the details of Emily's hypothetical learning trajectories in order to discuss her beliefs about student learning, and the changes in those beliefs that occurred over the course of the summer.

### **Lesson Structure**

Emily consistently utilized a teacher-centric lesson structure. That is not to say that she ignored student thinking, but that her lessons were principally focused on the teacher lecturing and posing questions from the front of the room. She expected students to draw on existing knowledge and prior experience at the outset of a lesson in order to grasp the purpose of the task, then to follow her example for solving simple examples. She then expected their understanding to proceed sequentially as they applied a general procedure in increasingly sophisticated ways to solve increasingly sophisticated problems. For example, when she first outlined her lesson on solving a system of linear equations, she decided to begin by "think[ing] about what else have we seen that we could build on to get there" – activating some previously-developed understanding. Drawing on their knowledge of solving linear equations, she would ask them, "how could it possibly be that we have two equations and they both have to be true?" She hypothetically introduced the lesson with the following comments:

How could it possibly be that we have two equations and they both have to be true? Not just one is true and not just that we're finding one answer, but we're finding a relationship between  $x$  and  $y$  that makes two things true. So there's a few ways, over the course of a

unit, that we're going to look at how we could approach this [...] let's just do them one at a time.

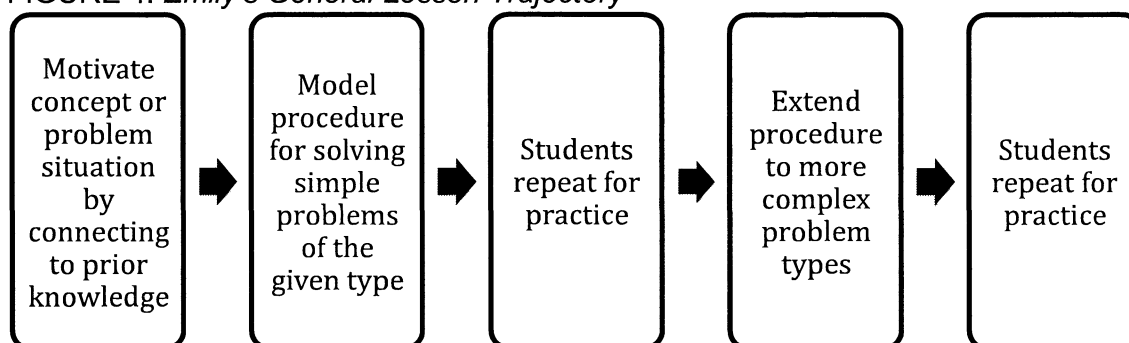
This presentation was designed to motivate the system of equations concept, and took the form of a teacher's explanation. She then proposed to guide students toward the substitution method, because, as she said, "I would like for my own organization to choose the method for them." This was born out of a belief "that students feel really, like, nervous about things being too open-ended" even though they are hesitant to be "boxed into" a method with which they were not comfortable. She described the subsequent class discussion as follows:

In the beginning it would be very concrete and very structured and I would choose very carefully the way that it [the problem] looks and also the way that we solve it [...] so there needs to be a really fine balance of "you have these choices, but if you don't really want to delve into uncharted territory, just follow this procedure, and then as you keep going and building on that, then it becomes more like we're making connections between things.

The basic trajectory, then, was for the teacher to propose a new type of problem to solve, discuss what it means, then introduce a procedure for solving it, starting with her "doing everything for them, modeling it, [with students] watch[ing] what I'm doing and why I'm doing it." The students could then copy this procedure on similar examples. After students worked on their own, she would apply this procedure to more complex examples, followed by students practicing similar applications. In her first lesson, those examples moved from two equations in point-intercept form to two equations in standard linear form. She summed up the trajectory as follows: "it starts out kind of open ended, then focus it, then they're practicing."

Similarly, the second lesson she developed, on defining *function*, followed the same trajectory. She proposed to begin by asking “when and in what context have they ever heard the word ‘function’ before”, then providing a definition at the board for the class to use. Finally, examples on the board would become increasingly complex, starting with a “function machine” drawing and moving to verbal directions such as “add five” and “divide 6 by the input”. Procedures would be given for dealing with each one, culminating in finding the domain and range of each of the examples, and these terms would be defined at the board for the class. Thus, during the first interview, Emily identified the general lesson trajectory shown in Figure 4:

FIGURE 4: *Emily’s General Lesson Trajectory*



She eventually summed up this general structure in the third interview:

We'll do one where I'm pretty much leading the way, maybe getting a little bit of feedback. Give another one that's similar and have them be more, you know, the drivers of this whole process, but really just repeating what I just did. Give them a couple chances to practice. And then just sort of start that process over again.

In the second interview, Emily made minimal changes to her lesson plans. For the lesson on solving a system of linear equations, she identified more specific example problems that she might use, but the trajectory of the lesson remained the same. Her discussion of how she would guide them to the

substitution method, however, showed a bit more attention and responsiveness to the thinking and understanding of individual students:

Guiding them toward the substitution method can be sort of artful because if I'm seeking all their ideas and their responses or their ways of solving it, they may not have thought of that. [They] might have thought of something else, they might have thought of that in an incorrect way.

However, she maintained her emphasis on teacher control of the lesson, stating, "It's just important [...] to validate what they're saying, but be like 'this is how we're doing it today, so just go along with it.'" She was clearly more comfortable with the teacher being in control of the direction and trajectory of learning, and, as I shall show below, interpreted her summer experience in terms of that. Similarly, the changes she suggested for the second lesson were minimal, consisting of adding an example of a function that was not one-to-one (she suggested " $x$  squared"), and deciding that the lesson was only a short introduction rather than a full class period.

She also made no substantial changes during the third interview, and reaffirmed the general structure wherein she would model the solution methods and encourage students to "try to match up step by step". She acknowledged that students aren't "always successful at executing that," attributing their failure to a lack of proper study habits or maturity. Despite this, the hypothetical learning trajectories that she identified for "typical" students in her classroom (discussed in detail below) consisted of students patterning their solution methods after those she demonstrated in class.

Emily did not point out any changes in her practice, describing her lesson structures, classroom organization, and teaching philosophy in consistent ways across all three interviews. Furthermore, those aspects of her own learning over the course of the summer that she identified as important were, for the most part, similar to the principles of student learning that were significant to her. That is, it would seem that her conceptions of student learning affected her summer experience more than they were affected by it. Those structures, such as focusing on procedural fluency and drawing on familiar concepts, that were important for her practice naturally attracted her attention during the program. Furthermore, the long-term exploration and “struggle”, which she found to be valuable but difficult, was rejected on the basis that it wasn’t appropriate for students.

However, “experimenting” with problems until connections emerged also played a role in her learning, and this was not something she described as a part of her teaching practice. In fact, she believed her philosophy to be very different from “experimenting”. However, it seems that she had incorporated more of it than she realized. While not as open ended as the summer problem sets, the lesson I observed prior to the third interview nevertheless emphasized completing several different examples in order to see connections, differing slightly from her previous descriptions. As she summarized it:

We spent a while solving different kinds of equations, and [...] at first it seemed like they were really different, like they had nothing to do with each other, you have to memorize this procedure, memorize that procedure and then we came to a point where I was like, trying to make it connected.

Her lesson focused on looking at several different types of problems, memorizing procedures for solving them, and then finding connections. It focused much more on repeating and mastering procedures than did the RLE summer program, but the idea of patterns and generalities emerging from multiple problems is similar. When asked about this, she made it clear that it was not a conscious effort to parallel the summer, and did not “put that together” (the similarity in structures) until discussing it during the third interview. This small shift in emphasis is consistent with the small shifts in beliefs indicated by the belief survey. Thus, Emily, seemingly subconsciously, incorporated some aspects of her own learning experience into her lesson structures. A similar, albeit small, shift in emphasis was also observed as she discussed student learning through her lessons, even though those discussions also remained largely the same.

### **Hypothetical Learning Trajectories**

In the first interview, when asked to describe the thought process of a “typical” student participating in the lesson, Emily again emphasized that students would be looking to her, as the teacher, to direct the lesson, and would be willing to follow her lead. She said that students would be “following the steps and trying to match back, like, the examples that they already have in their notes [...] and as we progress through, or as we’re introducing new ideas, to be starting to think a little bit bigger.” However, her discussion of student learning returned to the responsibility of the teacher to lead the discussion: “So that’s again on me to choose the examples to sort of illustrate some of these thoughts that I would be having and also to model that to be helping them to see my own thinking.”

In the second interview, she reaffirmed that students would learn through “the steps of the procedure that they’re following.” In addition, “on a higher level, think about when you get to that answer at the end, what is that telling you?” So, in addition to following the models provided, she expected some additional student questioning, some additional seeking of underlying concepts. Indeed, she made explicit the notion of student learning that had been implicit in her prior discussions on the topic: “They would be thinking about more procedural stuff and then the more conceptual and kind of go back and forth depending on what point they are in the problem.” That is, she believed that procedural understanding preceded conceptual understanding, and that the path to understanding began with mastering procedures. In the third interview, discussing the defining *function* lesson, she again noted that understanding comes about as the result of imitating, then mastering procedures:

I think that it’s pretty intuitive once you see a few examples that you’ll be able to follow that pattern and think about ‘all right, every time I write  $f(x)$ , that means I’m about to plug something into something. And then whatever comes after the equals is what I’m plugging into.” [...] So there could be something else, like, embedded within one of these problems that is a problem in and of itself, but the idea of a function wouldn’t be a problem.

In this excerpt, she indicates that the concept of function will emerge from completing and examining multiple problems. In other words, that conceptual understanding will emerge from procedural understanding.

Her emphasis on mastering procedural skills fits into Ernest’s (1989) *child’s mastery of skills model*. Ernest’s models of learning mathematics were organized around two constructs: a view of learning as active construction or



passive reception and a view of the student as autonomously pursuing his or her own interests or as submissive. A teacher utilizing the *child's mastery of skills model* views learning mathematics as more passive reception than active construction, and views students as midway between autonomous and submissive. Her lesson structures also contained elements of Ernest's *child's linear progress through curricular scheme model*, which views students as more passive than the previous model. However, Emily demonstrated attention to students' individual reasoning and motivations. Since she viewed the teacher as the leader of the classroom, students in her view were not solely pursuers of their own interests, but she did not view them as totally passive agents, either.

Her definition of a successful lesson was also based on mastery of skills. Specifically, she said she would consider the first lesson to be successful if "as I'm asking from the front or as I walk around seeing their work, I see that a majority of people are getting a majority of the steps of the problem" and that students can "solve a system of equations that's set up in this way, where it's  $y$  equals something,  $y$  equals something, and then you put it together and solve for  $x$  and  $y$ ." Regarding the lesson on defining *function*, she stated:

I would find a successful lesson to be, could they evaluate a function, or could, if I gave them an input, could they tell me an output. If I gave them a domain, would they know what that means. Just, like, a usage kind of thing.

In interview three, she again expressed that success would be defined in terms of execution, saying, "as I'm circulating around, I'll give them, you know, a handful of examples, and I'll see how they execute it."

Emily's hypothetical learning trajectories were based on the teacher conveying procedures to students. Though her lessons were not strictly lecture, she believed the teacher should control the goals and directions of the course, with a focus on modeling proper procedures. She expected that, by practicing and mastering these procedures, students would develop a conceptual understanding of the topic. However, her criteria for demonstrating that understanding was procedural fluency. These expressed beliefs were robust, consistent with the classroom organization and lesson structure observed during the school visit, and were not significantly impacted by her summer experience.

However, despite the overall consistency, one observation during the classroom visit indicated that some aspect of her conceptions regarding student learning may have changed. During the lesson, while working various examples on the board, Emily repeatedly asked students, "what does your instinct tell you?" When I asked her about this, she responded by noting it had developed naturally over the course of the semester:

Well, after we've spent a while solving different kinds of equations, [...] at first it seemed like they were really different, like they had nothing to do with each other – you had to memorize this procedure, memorize that procedure. And then we came to a point where I was trying to make it connected, like "we're trying to do this same thing every time, but we have some small difference." So people would say things like "I don't know how to get started at all." So if I make one small change, then suddenly they knew what to do [...] So somehow to have some kind of gut reaction to "oh, I wish I could do this. Well, ok, how could I make that happen?" That's sort of the thought process that we've been building up to."

So, rather than providing new procedures each time, she was trying to help students to generalize by appealing to their intuitions about what kind of changes

would be desirable to make a problem like something familiar. This conceptual emphasis was striking given its inconsistency with our discussions during the interview, especially considering how consistent the observed lessons were with the constructed lessons otherwise. Taken with the small change in lesson structure described above (more emphasis on seeing connections among multiple examples), Emily's change in emphasis hints that her summer experience may have had some impact on her beliefs.

However, the fact that these changes occurred in the midst of a general consistency indicates that Emily's beliefs and her summer experience each impacted the other. That is, while her summer experience played a role in shaping her beliefs, her beliefs also played a role in shaping her summer experience. Next, I will discuss Emily's conceptions of mathematics as a discipline, highlighting how her time in the summer program impacted them, and how these conceptions interacted with her conceptions of student learning.

### **Emily's Conceptions of the Relationship Between School and Research Mathematics**

As mentioned above, Emily had an undergraduate background in mathematics, so her preparation and experience were somewhat more advanced than many other participants. Still, her conceptions of teaching and learning mathematics were largely procedurally-based. When asked in the second interview how she would describe mathematics research to a student, she responded by saying:

Before this summer, I think I knew even less about that, and I still don't feel like I have a full accurate answer to that question [...] but

in terms of the pure math that we've been doing, in terms of number theory, [...] you just take maybe the ideas that other people come up with or just other things that you've been exposed to [...] and just playing around with it and seeing what you could come up with. Which, I think that the analogy toward, from this to other science fields is a good way to explain it. Where you could say, like, imagine that you're a chemist and you've seen that these chemicals have certain relationships, but you're wondering what if you introduce this other idea. So you go into the lab and you start mixing things together and you write down what you did, and see what the results were. Is that what you expected? Oh, you adjust what you thought you should do, you adjust the measurements [...] so you could say you do the same thing with some numerical idea or even with things that we've worked with before.

In this quote, she expressed her apprehension about describing mathematics research, and then draws a parallel between mathematics research and scientific experimentation. Such an analogy had been used earlier in the summer by the lecturer. Note, though, that she refers to building on the ideas of others rather than creating them oneself. Consistent with her adherence to this description, Emily believed that the summer program was indeed a model of mathematics research. Speaking of a specific set of problems she had worked on, she said, "I felt like it was, like, a true experiment and then you come up with these conjectures and then you keep working on them." However, she noted that her undergraduate experience was inconsistent with the definition of mathematics research that she chose to adopt:

I was a math major, but I never did any original research. And I definitely had experience being presented with a theorem and then being expected to prove it, so I wrote a lot of proofs, but I knew that they were true statements, so it wasn't like I was questioning "I wonder what's going to happen if I try this?"

Here she highlighted an important aspect of mathematical exploration that she felt was absent from her own education – the opportunity to conjecture and

determine the truth of those conjectures. Her own experiences with proofs consisted of proving statements that she knew to be true.

Since her previous mathematical experience had not modeled mathematics research, she identified the nature of mathematics research as one thing she had learned during the summer. She said she had learned “this idea of exploration and experience before kind of like revealing what the answer is or what the statement is.” She was hesitant, though, to see that type of long-term exploration as something that her students could do, saying, “even if I had my students like experiment with something and try it out, I would have never had them struggle through something for as many days as we did.” This perceived disconnect between mathematics research and student learning was reflected in her response to two specific belief survey items that were read to her. As with the other interview subjects, I asked her to respond to the statement *when mathematicians do mathematics, they are doing something fundamentally different than when students do mathematics*. Emily said, “I think that’s true [... but] I don’t necessarily want that to be true.” She explained:

I think any high school student that I’ve worked with has seen it like “this is already in place – tell me what to do and I’ll do it.” And hopefully they’re making sense out of it as they go, hopefully they’re exploring it or making connections or hopefully are at least able to guide them toward making connections, but I don’t think that they see it as something new.

Thus, she believed that, just as her undergraduate experience differed from mathematics research because it consisted of proving statements that were known to be true, school mathematics differed from research because it was focused on new results, while high school students were very aware that the

mathematics they were learning was well-established and their own work played no role in shaping it. However, when asked to respond to the statement *the thought processes involved in learning high school mathematics and the thought processes involved in researching mathematics are the same*, she decided that these two are “the same”:

I do think that whether you're doing original research or you're working through a textbook that's very scripted and formulaic, you're still kind of going back to some set of rules that you understand or some set of patterns that you understand or some concept that you're trying to apply. Whether it's in a new way or practicing over and over the same thing, you're still going back to some understanding that has already been established.

Thus, even though she thought that students and mathematicians were doing different things, she believed the thought processes to be similar. Note, though, that she characterized these thought processes as referring to rules, procedures, and patterns that one would apply. She believed the thought processes to be the same, but believed them to be the same as the procedure-based hypothetical learning trajectories that shaped her practice.

The way Emily viewed the relationship between mathematics research and student learning is indicative of the way her summer experience shaped her beliefs. However, it also shows the ways in which her existing conceptions influenced the way she interpreted that experience, thereby limiting the impact the program had on her beliefs. Below, her time in the summer program is discussed in detail in order to explain the changes discussed above and to demonstrate how her conceptions shaped her experience.

## **Discussion and Conclusion**

Emily's case points out that belief systems are not just entities that are affected by experience, but also mechanisms for interpreting and understanding those experiences. Her conceptions of how learning proceeds were shaped by her own experience as a mathematics undergraduate major and as a beginning teacher who developed her teaching philosophy largely through trial and error. Primary beliefs are often formed during one's time as student (Clark and Person, 1986; Thompson, 1992) and frequently have emotion-packed experiences or cultural transmission as a source (Ambrose, 2004). Certainly Emily's early teaching experiences could have been an emotion-packed experience (her characterization of that time as "anything goes" and herself as having "no clue" lend support to this idea). Her core beliefs in procedural understanding preceding conceptual understanding and in the teacher as leader, facilitator, and transmitter of knowledge were strongly enough held that they could be considered knowledge for her, and therefore non-negotiable. Just as a mathematical concept is difficult to tease apart once an individual has constructed it (Cobb, 1989), it seems that once Emily constructed her knowledge about student learning, it was difficult to envision any learning scenario without it. Thus, her conceptions shaped the way she viewed her own learning experience. She saw more structure and identified more procedure in the summer program than other interview subjects did as the result of her conception that learning proceeds through those mechanisms.

Nevertheless, two small and subtle changes occurred in her practice – subtle enough that she did not initially recognize them herself until she was asked to reflect upon them. First, while her lesson trajectory did not change, she began to emphasize patterns “emerging” from the multiple examples that she used as part of the trajectory. Secondly, she began to appeal to students’ “instinct” in order to see patterns and generalizations. Both of these changes mirrored aspects of her summer learning experience, indicating that she internalized that experience and drew parallels between her own learning and that of her students. The fact that she was unaware of these changes until reflecting upon them indicates that they had been seamlessly integrated into her existing conceptual schemes. The exploration she constructed for her students was sufficiently scaffolded so as to not present a conflict with her ideas of how class should proceed.

Furthermore, her conceptions of the mathematics research process were shaped by her conceptions of learning in general. Her characterization of mathematics research was rooted in the same aspects of student learning that were core parts of her beliefs. She saw the work of research and student learning as different, but the thought processes as similar. This was partly because the open-ended, exploratory nature of mathematics research (see Muir, 1996) was at odds with her own conceptions, but the thought processes she identified as part of mathematics research, when filtered through her own conceptions, were consistent with her ideas about student learning. Thus, her conceptions of mathematics research were shaped by her existing knowledge



just as her conceptions of student learning were. She did seem to construct some parallels between research and learning, but only to the extent that both were shaped by her existing conceptions and personal learning experiences.

As a first-summer participant, Emily's work was not as explicitly "research-like" as that of the second-summer participants. The changes to her conceptions were correspondingly small, as the experience was familiar enough to be accommodated by her existing conceptual schemes. Nevertheless, the opportunity to learn through exploration and "pattern-sniffing" (Cuoco, Goldenberg, & Mark, 1996) did have an effect on her work, though that effect appeared to have been subconscious.

### **Deborah**

#### **Background**

Deborah was a first-summer participant in the RLE program, and was one of the most experienced teachers in the program with sixteen years of teaching experience prior to starting the program. She had started working as a teacher immediately upon graduation from college, where she earned a degree in mathematics with an education minor. Since she had studied some science as well, she was dual certified in mathematic and earth science. After teaching for four years, she moved on to other pursuits before returning to teaching after a long absence, and taught for two years before transferring to the vocational/technical school at which she was teaching upon arriving at the RLE program. She had been there for ten years, and said, "I love it [at the school]. I'm going to retire there." After her first year at her current school, she had

earned National Board Certification, and she identified that process as a formative and influential one.

Students at Deborah's school spent half of each school day working in one of several practical laboratories on campus and half in academic courses. Deborah taught mathematics and occasional science courses in the academic program. Over the past few years she had "primarily been teaching the at risk kids, the kids who have failed the [state standardized] test" and said she "love[d]" teaching that group of students. For the past several years, she had worked with a special education teacher, team-teaching remediation courses for students who failed the state standardized test. These courses had focused primarily on algebra and geometry, with "a little bit of statistics, probability". They had worked to make special arrangements for their remedial courses:

We really worked hard at lobbying for what we thought we needed. That was to have our kids for more than one year so that the majority, the core of those kids – as soon as we could boot a kid out into the mainstream, we would do that – but keep the core of kids for three years, to get them through the [state standardized] test [...] Now all our kids start, tend to pass. By sophomore year we only may have a couple, a handful who don't pass. [...] We also lobbied to have them a double period. We have a four period, four day rotation so we would [...] only have 40 minute periods, which is way too short. So we would have 40,80,40,80, so we had extra time with them.

However, this arrangement, of which she was a very big proponent, had recently changed. The state department of education "said that [the school] can no longer have those essentially separate classes." Thus, she no longer had class periods that were 80 minutes long, and she said she was "frightened of what might happen [...] it really changed the development of a community [...] by

changing that one factor, it's difficult to build on a real learning community for math." She still valued classroom community, and continued to teach the same students, but believed the new arrangement to be less than optimal. Partly because of the change in the way her remedial courses were structured, she was teaching an upper level geometry course the year after her first summer in the RLE program.

A variety of prior experiences had a hand in shaping Deborah's conceptions of learning and teaching. As a mathematics student, she described her program thusly:

It was all theoretical, all the way through [,,,] I latched onto earth science as something that I really liked and cared about, so when I studied crystallography, crystallography makes use of linear algebra, matrices, and it was like, it clicked! (laughs) You know, like all of a sudden everything started to make sense in a much better way for me, that there was this connectedness between things that I'm studying and things in the real world. I'm also a very visual math person. I'm more toward the geometries rather than the algebras. So, I became a statistician, so very applied. Always, I think, from that moment, math became very applied for me. The meaning in math for me was in its applications.

So, in her own experience, she appreciated mathematics that could be represented visually and that which could be applied, connected to other disciplines. Furthermore, Deborah had participated in some professional development programs that she identified as especially meaningful, partly because they required reflection and introspection. She had recently participated in a "lesson study community in secondary mathematics". She led an interdisciplinary team of teachers from her school, and their goal "was to

integrate math and the [technical disciplines] using lesson study.” She described lesson study as being

about thinking about how your students are learning – is what you’re doing, are the students getting out of what you are doing with them what you thought they would? What it does for you is it puts that little extra column in your lesson plan about what are those students’ responses that you expected. What are you expecting the student response to be to this, this, this, and this? So it changes the way you think about your lesson. You also realize that a lesson’s never finished and that the class in front of you changes the lesson.<sup>24</sup>

Thus, she valued and had consciously worked on considering how she expected student learning to proceed through a lesson. She characterized the experience as “extraordinary”, and appreciated the community of teachers that developed, as well as the integration of subject areas that resulted from their cooperation. She had also twice participated in SummerMath for Teachers (see Schifter & Fosnot, 1993; Simon & Schifter, 1991), a mathematics immersion professional development program that uses high school mathematics content and encourages reflection through journaling. Partly because of these experiences and the introspection they required, she felt she had a “very well-thought-out perspective” on her work and her students. She said her perspective came about as the result of:

a lot of hard work on my part [...] I don’t know what makes that happen for a teacher [...] It happened for me when I returned to teaching after a 20 year absence. It happened for me when I had the most difficult, the most needy students that I had ever had in my life [...] and when I made the decision to pursue National Board

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<sup>24</sup> For a fuller description of lesson study, see Stigler & Hiebert (1999).

Certification. That year of introspection changed me, changed me as a teacher.

Furthermore, she was comfortable and confident with her perspective, saying, “I’m very comfortable with my beliefs about teaching about what happens in my classroom, about what I do, don’t do, make decisions and change.” She also said that she did not think that her comfort in her own beliefs developed until she participated in SummerMath and, more importantly, the National Board certification process.

She thus came to the RLE summer program because she was interested in learning more mathematics and in developing her perspective on teaching and learning mathematics. “I’m always looking for meaningful experience,” she said, “that’s why I’m here.” As described below, her previous experiences as a student and as a participant in professional development experiences shaped her conceptions of teaching and learning, which impacted the way she interpreted her own summer learning experience in the RLE program. The topics of visual representations, applications, community, and reflection played important roles in both her conceptions of learning and her interpretation of her experience in the RLE program. That summer experience is described in detail in the next.

### **Deborah’s Summer Experience**

Deborah said that she enjoyed two things about the RLE program: “Doing math and doing math with other people.” This section will unpack those two aspects of her experience and consider how her existing conceptions played a role in shaping that experience.

Just a few days into the summer, Deborah described the program as “like going back to college” and as “good exercise for me because I’ve been away from this for a long time [...] this is keeping my brain happy.” Feeling her knowledge and abilities stretched helped her “remember how my students feel.” Indeed, she thought this was a significant benefit of the program because she valued the opportunity to “go back and understand how your students feel” in any learning situation. Like other teachers, she was challenged by the program’s organizational principle where problems were presented in problem sets before the material was completely described in lecture. At the end of the summer, she described it as “painful” and said, “you feel that discomfort – yeah, it’s a pretty direct hit”. However, consistent with her philosophy that experience learning as a student is a beneficial, she said, “but that’s ok [...] it’s ok to feel that pain. I mean, I feel like I’m sure most of my students feel like when they step in my class the first day (laughs).” She even said she hoped “all of those teachers [in the program feel that.” When asked why, she responded:

Because I think you need to remember how that feels and not get upset at the student who says “I don’t get it. I don’t understand this, I don’t see it at all,” and you have to stand on your head five different ways to make them maybe see it (laughs) [...] I think it’s reinforced how I think about my students. I think it’s definitely something I’ve thought about a lot [...] it’s hard for students when there’s a mismatch of learning style with the teacher’s style. (pauses) To recognize it in yourself is the first step to recognizing it in your students.

Thus, her first summer at the RLE program strengthened her existing belief that experiencing learning for oneself was the cornerstone of understanding, appreciating and empathizing with students’ experience. Furthermore, it allowed

her to actually feel what it was like to be a student, and thus increased her sensitivity to students.

Though struggling without clear direction was frustrating for her, Deborah felt that her “maturity” gave her perspective on it, that she understood “it isn’t all going to happen right now, and there’s no reason to rush! [...] I don’t feel that pressure.” Perhaps in part because of that perspective, she determined that gaining experience before seeing theorems could be valuable – that “experience” and the struggle to understand prior to direct instruction might be beneficial to learners, saying:

I guess I’ve come to appreciate the experience more – appreciate that you need to do the numericals! (laughs) You need to! I mean, in spite of what they are saying, “do the proofs, do the proofs,” the proofs don’t come as easily or as obviously without the numericals behind them. So this idea of building and appreciating the history and understanding from the lecture, [that] years went by between changes in thought or discovery of structure. The idea of looking for structure, I’m sure I must have had it at some point, but I’m sure I lost sight of it a long time ago from being in the classroom [...] If anything, being here has reawakened that remembrance of what the heart of mathematics is – where it comes from. People work at these things a long time! (laughs)

In other words, having experienced exploration for the purpose of revealing underlying structure, she came “to appreciate” this as an important aspect of mathematical work. Interestingly, this quote indicates that she came to appreciate exploration not just as a tool for learning, but also as the mechanism by which mathematics has developed throughout history. Furthermore, she saw the latter fact, that exploration has been important in the history of mathematics research, as the primary reason for exploration’s significance, more so, even, than the fact that it was beneficial to her own learning or would be beneficial for

her students. Implicit in her emphasis on mathematics history is an assumption that the historical development of mathematics is a useful and appropriate model for mathematical knowledge development in general.

Deborah mentioned “doing math” as one of the pleasures of her participation. The work stretched and challenged her, but she saw that as valuable. She appreciated that struggling to understand the structure of mathematics before being told about that structure was challenging but beneficial for her own learning, and also characterized it as an important aspect of mathematical development historically. Doing mathematics helped her understand what it felt like to learn and struggle, but it also caused her to reevaluate her conceptions of how mathematical knowledge develops. Furthermore, her discussions of her experience “doing math” illuminated a belief that mathematics research (mathematical discovery) parallels at least one potential learning process. This will be explored further later in the section.

In addition, Deborah said that she enjoyed “doing math with other people.” As described above, developing a community of learners was important for Deborah in her classroom as well as her professional development experiences. Perhaps in part because of a predisposition toward its importance, she noted the social and community aspects of the RLE program, saying that it gave her “some insight about grouping and a little bit more about group dynamics than I think I’ve learned other places.” When prompted to expound on that statement, she said:

[I learned] that there’s something to be said for letting group relationships mature over a period of time [...] teachers are always looking for different ways of grouping [...] but I think I’ve learned more about differentiated instruction, and groups, and that having a



group who work at about the same pace can be important. When the ideas are deep enough that you need to linger, I think that working with people who are at your pace is important. Or it's helpful, but I don't know that that's something I really felt before.

The working groups for the first year teachers were not assigned, but simply developed as teachers sat together and got to know each other. Deborah moved among groups a bit, partly because of her own dissatisfaction and partly at the suggestion of her grader. Her discussions about group work indicated that her experience in the RLE program led her to believe that, with concepts that are "deep enough that you need to linger", it was important to work with a group of peers who had similar ability levels and complemented each other by working at a similar pace. That is, in her experience, a group with homogenous ability levels was preferable to a heterogeneous one.

Finally, at the beginning of the summer, Deborah had identified introspection and reflection as a very important mechanism in her growth as a teacher. Her comments above indicate that she was reflecting on her own experience, but she did not feel that it was built into the program: "[Reflection] is something that I carry over to my work here, but it's not something that I see intrinsic here yet for everybody. But I'm not done yet." She wondered whether the one-day seminars that met over the course of the ensuing school year might initiate more reflection. Regardless, introspection had become such an important part of her own work (she said "I've just become that kind of person now") that she consistently reflected on how the summer program related to and impacted her beliefs and teaching philosophy. Below, I will discuss her beliefs about student learning and the nature of mathematics, detailing some of her reflections

and some of the ways her conceptions were impacted by her summer experience.

### **Belief Surveys**

Deborah's belief survey response scores indicated that she believed students constructed their own knowledge rather than receiving it from others (category I). However, her agreement with that belief was only mild on the pretest, with a mean response score of 3.667 (3 being "neutral"), and her posttest score was the same. Her responses for category II and III also showed no change, scoring 3.5 and 3.625, respectively, for both administrations of the survey. These scores indicate a belief (though not a strong belief) that the mathematics research process and the process of student learning in mathematics are similar and that students are capable of significant independent mathematical insights without teacher intervention. Her scores are summarized in Table 6:

TABLE 6: *Deborah's Belief Survey Response Scores*

	Adjusted Category I	Adjusted Category II	Adjusted Category III
Pre	3.667	3.5	3.625
Post	3.667	3.5	3.625

Deborah's belief system was well-developed prior to her arrival at the RLE program, and she was confident in her philosophy of teaching and learning while leaving room for it to be impacted by new knowledge and experiences. Thus, it seems that her belief system was either robust enough to withstand any challenges presented by her summer experience, or that experience was

consistent enough with her existing beliefs to be incorporated into them. The only category showing any interesting change, category II, concerned the relationships between mathematics research and student learning. As described in the section above, exploration for the sake of finding structure emerged for Deborah as a dominant theme for both learning and research, and this could have played some role in shifting those beliefs. In the sections to follow, her beliefs about student learning as they emerged through the lesson planning tasks, and then about mathematics research, will be discussed in detail.

### **Hypothetical Learning Trajectories**

For Deborah, lesson planning included a conscious effort to outline how she expected students to respond to the activities that made up the lesson. Thus, her discussions about lesson planning were inextricably intertwined with the learning trajectories that she expected from her students. The discussion contained in this section highlights all parts of her hypothetical learning trajectories together in order to accurately reflect the way Deborah herself engaged in the tasks and her work as a teacher.

Many of Deborah's students struggled with mathematics, and were in her class because they had failed the state standardized test in math. She described her attitude toward her students as follows during the first interview:

I take nothing for granted, I make no assumptions about what you know and don't know [...] We really, my partner and I really strongly believe that, for most of these kids, the root is not their understanding of mathematics, but their understanding of language. And it's the language of mathematics that is hard for them, not the mathematics itself.

Though she acknowledged that some of her students had underlying learning disabilities, such as difficulties with “long term memory recall and that sort of thing”, she felt that language issues were the root of most of her students’ problems with mathematics. She also believed that students “could get many of the concepts” if the language burden or the arithmetic burden was lifted in some way. Thus, she approached her lesson planning with the assumption that students needed long-term, in-depth experience with mathematical language, notation, and communication in order to increase their comfort level and make the language routine. Furthermore, lessons needed to begin with concepts and notations that students could understand so that they might be eased into deeper concepts. In fact, she eschewed textbooks in her classroom because “they’re too hard to read” – students were intimidated by them and therefore become overly reliant on the teacher, “reliant on not thinking for themselves.”

Deborah’s first lesson was on solving linear equations, and, in the end, the lesson trajectory she described was a multi-day, perhaps multi-week, unit. In the first interview, when she initially planned the lesson, she noted that building confidence would be the first obstacle to overcome, saying,

The first thing that I would need to do is convince the kids that they could do this. And probably an approach would be to do something that they can do in their heads. So I wouldn’t start with a coefficient other than one in front of the  $x$ . And it wouldn’t have subtraction, it would be addition, so we’re just looking at a very simple form of this. But convincing students that they can actually do what I’m asking them to do [is the first goal].

Thus, her lesson began with a very simple problem that required no physical manipulation. Furthermore, she thought it important to “put it in context”. That is,

to have the problem be part of some word problem, or real-life situation “rather than just coming from an abstract.” She believed her students “have a lot of trouble understanding what a variable is,” so she proposed to use multiple representations of the equations and model their solutions in a variety of ways. One of the first models she proposed to use was Algeblocks<sup>25</sup>, which are three-dimensional blocks that can be used to physically represent algebraic expressions. She used these often in her classroom, and was actually utilizing them when I observed her class. She liked to use them because they were “concrete” and provided a model for students, which offered the opportunity to make “connections between the physical and abstract.” She said,

I think the kids need connections, and it really doesn't matter what you use for models, as long as there's a model [...] something like the blocks are sufficient as long as there's a drawing, something physical [...] it doesn't matter what it is as long as there's something to connect to, to get you from the concrete into the abstract.

In general, she believed student understanding moved from concrete to abstract representations, and her lessons reflected that. She described the process thusly: “I would do blocks, describing the blocks, and then, the third step would be to add the abstract notation. The notation comes last.” So, after modeling simple equations with algeblocks, student would be required to describe the algeblock models in words and drawings. Finally, they would move to representing equations with variables and standard notations. As she put it, “the goal is to get them off the blocks either to be making pictures, pictures and

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<sup>25</sup> Manufactured by ETA/Cuisenaire.

language, or just language. At some point hope that there's just the symbolic language and they can manipulate that." In other words, she said, "I start with a very specific format. And then that format gets, over a period of days, gets less and less rigorous, more open-ended for them."

Simultaneously, the problems students would be asked to model and solve would become more complex "step by step", incorporating negative coefficients and constant terms, then multiple appearances of the variable. However, she felt it was important that her students become comfortable with each new complication before more were added. She claimed something as simple as moving the variable to the right side would make students react "like it's something they've never seen before [...]" for some of my students, that's just such a huge bridge for them to cross," and when variables appear on both sides of the equation, "all hell breaks loose". As students encountered various problem types, she proposed that it was important to "teach that this [problem solving] is an art form." In particular, she wanted to communicate to students that

you make choices as you go along the path of solving any kind of equation, and that choice, [those] choices that you make can sometimes make it more difficult, make it easier to solve. So you have to learn to make good choices, you have to experience what a choice is and what's not a good choice and why.

Thus, she viewed equation-solving as an "artful" process, noting that there is "not one way" to solve equations. She felt that communicating the artfulness of choice-making was an important goal of the lesson, particularly because, when they later apply their knowledge "no one is going to be there," so students need to be enabled to make decisions independently. She proposed to accomplish

that goal by validating students' choices and allowing them to see if those choices were helpful or not. She said, "you have to always be accepting of whatever way a student has to show you how to think about things [...] The day you become a better listener is the day you're a better teacher."

Deborah's first lesson, as originally constructed, emphasized students developing flexible understanding of and ways of solving linear equations as a primary goal. She believed this happened by starting with concrete representations (in this case, Algeblocks) and slowly transitioning to abstract representations (standard notations) through increasingly less concrete representations (drawings and verbal descriptions). She noted that the notation would be introduced "only as needed [...] it will be dependant class to class, individual to individual", so the specific nature of the process was a response to the needs of her individual students and unique classes. At the same time, she wanted students to solve increasingly complex examples, with complications (such as negative coefficients and multiple appearances of the variable) appearing one at a time. At the end of the unit, she had in the past done an "equation marathon" with her students, where they solve as many equations as they can, with approximately 20 that everyone will solve, over the course of two days. There were prizes for winners, and she chose this activity as the final wrap-up of her lesson. She said that she would call the lesson successful if she could see "each student give me an example and solve it [...] create a problem."

In the second interview, Deborah made a few changes to the lesson. First, she decided to add more "do in your head" problems at the beginning to

serve as motivation for the unit. “Not necessarily word problems, they can just be examples. They would start with word problems [...] let’s just do a whole bunch of these together in our heads.” By exploring multiple examples, students would be introduced to the material, gain “mental math practice”, and build confidence in their knowledge and ability. Secondly, she wanted to add “a discussion to motivate the symbolism in a little more concrete way [...] maybe demonstrating balance some way [...] something fun. Maybe even motivating with something from our campus.” She felt such a discussion was important because

it needs to be explicit. We need to understand there’s a need for some kind of symbolic language, some way to represent that unknown quantity. It can be a cup, or a box, or anything, and eventually it has to be a letter [...] but the whole idea still continues – to get increasingly more difficult.

Both of the above alterations were concerned with providing motivation for the content. A third change was designed to delve more deeply into the context of the lesson in pure mathematics. In her words:

After this [RLE program number theory] course, I might actually spend time talking just about natural numbers, integers. [...] it’s something they always test, always ask on the [state standardized] test, how our number system fits together. I don’t think they have a good feel for that vocabulary. This is certainly a place where how we build this, you could bring in a lot more about the history of our number system and how that all came into being.

Recall that her summer experience led Deborah to a revived appreciation for the history of mathematics and how the development of mathematical ideas might be useful for the classroom. The change to her lesson trajectory described in the



above quote mirrored her appreciation for mathematical history as a motivator and inspiration for student learning.

Despite these changes, the overall goals and the basic structure of her lesson remained the same during the second interview – student representations of solving equations moved from concrete to abstract as they tackled increasingly complex equations. Furthermore, her learning goals still focused on language. She expected that students participating in the lesson, through “experience; practice; oral, written, demonstrable practice,” would be developing “some comfort with symbolism” and a list of questions that would guide their decision-making as they solve equations. Similarly, when I visited her classroom for the third interview, she reaffirmed the general hypothetical learning trajectory she had constructed. She again emphasized that the overarching goal was to focus on language and refine student decision-making:

Developing the language [is important]. It's not enough just to be able to do, but to be able to say it, and to explain it verbally. I think these kids don't own it until they're able to do that. They'll always be looking for a model, always be looking for someone to get them started [...] without developing an understanding of that art form. [...] So I think that's the focus for me, that's the important thing, is the development of language – to be able to speak yourself through the problem.

Indeed, when I spoke with the special education teacher with whom Deborah often co-taught, he told me, unprompted, that “it's really all about the language,” echoing the philosophy Deborah had espoused repeatedly. In that third interview, she again identified language development as the primary learning goal as well as her students' principle obstacle to learning. She discussed student learning in terms of their development of mathematical language fluency,

saying, “I’m wanting them [her students] to gain a facility with the abstract language with representational thinking [...] To understand, I don’t know, the symbolism - to be able to translate either way.” Just as she had said before, she believed students achieve this through “lots of practice” and through the use of models that are initially more accessible than mathematical notation:

I hope things like the algebra blocks help them, [develop comfort with mathematical language] - connecting to something real, connecting to some kind of model. I don’t think it happens without [some model]. It doesn’t matter if they’re made up things like the algebra blocks as long as there’s something that they can attach it to. I think it helps them. Not all of them, but many of them.

Throughout all three interviews, Deborah’s focus on language acquisition remained constant, but she was unable to pinpoint the developmental steps that went along with it. She would only say that language was acquired through “practice”, meaning she felt there was some sort of enculturation or initiation process where students, through repeated exposure, began to feel more comfortable with abstract representations.

The second lesson Deborah developed was on defining the abstract notion of *function*, which required more specific thinking about student acquisition of abstract concepts. She reiterated that, for the function concept, as well, language acquisition was the significant hurdle that students had to overcome: “It’s the notation that gets in the way, it’s not the understanding of what a function is.” She identified the “underlying theme” to be “about correspondence and about understanding how a set maps into another set.” In the first interview, her hypothetical learning trajectory for this lesson began with a “splash”, which she described as a sheet with “a variety of different things on it,” all related to

functions. In groups, each student “has to pick something from the sheet and they have to either define it or make some comment about it that they can share. And they might find nothing there that they recognize or know, and so they can ask a question.” The goal of the activity was to assess and activate students’ prior knowledge, and to allow the class to generate ideas that could then serve as the starting point for the lesson. Thus, she wanted the lesson to be flexible enough to accommodate whatever might arise from the splash. She also suggested that she might have a sorting or matching activity where students categorize examples in terms of whether or not they are functions. The examples would be various representations, allowing students to “look at them, experience them, see what makes something a function and what makes something not a function.” Note that, though she did not emphasize exploration as a means for finding structure in the first lesson until the second interview, there were some elements of it present in her structuring of the second lesson even in interview 1. Furthermore, her use of the splash in order to generate the ideas that would then be used to guide the instruction was indicative of her belief in individualized student learning.

Deborah anticipated students constructing their understanding of function by experiencing multiple examples, repeatedly categorizing examples, and “through building a classroom culture where, you know, we all start coming together” (reaffirming her emphasis on the cultural and social dynamic of the classroom). For a formal definition, she drew upon visual representations, which she had earlier identified as important to her own learning. In defining function

for her (hypothetical) students, she said “for me, the visual idea of a vertical line, that it passes the vertical line test, I think is something that most of my kids can carry with them.” She anticipated that students would construct an understanding of the vertical line test by graphing multiple examples, “grinding graphs.” She did not view the introduction of notation and new ways of representing functions as terribly problematic for students, stating that they were just “a different name” for the same concept. Thus, her lesson started by prompting students to call upon their prior knowledge, allowing her to assess that knowledge and set the direction of the class appropriately. From there, she would move to a sorting activity that would give students the chance to experiment and explore until the definition started to emerge for them. The class would then collect its ideas in order to formulate a definition, for which she drew upon visual representations. The themes of language development, translating between representations, and group communication ran throughout the lesson.

During the second interview, Deborah did not make any changes to her lesson plan, reaffirming the value of the sorting activity by saying it gives them “a feel for function” and identifying “experience, practice” as the key element for student learning. In other words, experience with multiple examples would help students develop an intuition for the concept. Her discussion of the sorting activity revealed some of her beliefs about effective teaching and the nature of student learning. In particular, she again discussed the importance of allowing students the opportunity to make and justify decisions:

I like them [sorting activities] because the kids have choices.  
Sometimes they come up with things you haven't even thought of

(laughs), in terms of ways of grouping [...] I think it just gives rich conversation in the classroom, and that makes them justify their choices. It's kind of what a mathematician does, huh?

In terms of her espoused beliefs, this statement was very much consistent with her discussions from the first interview. She believed it important that students be encouraged to make choices and that the class discuss strategies for making good choices, and that groupwork and classroom dynamic are important factors in encouraging that type of learning. However, the above quote contains a reference to the work of a mathematician, which was not present in her earlier discussions. Recall that the RLE program helped her reconsider the relationships between mathematics research, particularly the historical development of mathematics, and the student learning process. Her new attention to mathematicians as she discussed how she expected students to learn is further indicative of a change in the way she thought about the two processes.

In the third interview, Deborah only made one change to her hypothetical learning trajectory for the lesson on defining *function*. She said that the group I observed struggled with math, and she therefore might change the splash because she did not “think any of them have ever heard the word *function* before.” However, she still believed in the sorting activity and in the general trajectory of student learning, and still emphasized the language of mathematics:

I would definitely do the sorting activity again. I don't know, I think sorts are really important. Being able to sort develops a way of observation, makes them keener, keener observers, if they have experience sorting things. It's not enough just to say “this is this, this is this, this is this.” That's not enough. They have to be able to

use it, work it [...] And being able to explain your choices is again, the whole language thing.

Deborah's hypothetical learning trajectories were *learner-focused* (Kuh & Ball, 1986), driven by her conceptions of how students learn. Her observation and belief that her students' struggles had their root in language learning drove her lesson planning, focusing both lessons on moving students from concrete examples to abstract representations. She emphasized decision-making and felt it was important to provide student opportunities to make both successful and unsuccessful decisions in order to learn through experience. She also believed discourse and a classroom culture that encouraged it were important, and consciously tried to develop such a culture in her classrooms. The conception that student understanding develops individually was one of Deborah's core belief, though while she paid attention to students' interest, she did not believe their learning proceeded autonomously. Rather, in her lessons, the teacher behaved as a guide for learning. Thus, her beliefs fell into what Ernest (1989) called the *child's constructed understanding and interest driven model*.

Her conceptions about the trajectory of student learning and her core teaching philosophy were highly developed when she arrived at the RLE program, and they did not change significantly over the course of the first summer. Indeed, because the program mirrored much of what she believed to be important, and because her own learning experience in it was often consistent with what she expected of her students, she said it "reinforced how I think about my students." However, slight changes among peripheral beliefs were observed. First, she espoused a new appreciation for exploration at the outset of a lesson in

order to find structure, so that understanding and intuition about a concept might emerge for students. Because exploration was an important aspect of her own learning experience, and because she also came to believe it was the key aspect of mathematics research, she began to reassess the relationship between mathematics research, particularly the historical development of mathematics, and student learning in school mathematics. She came to see the two as parallel processes.

The impact of Deborah's summer experience was most evident in a "top-level" geometry class that she taught the following year. She had not taught the course in "many years", and in the second interview indicated that she believed her experience would impact that class most of all. When I visited her at her school, she noted the geometry class as the aspect of her practice that was most impacted by the summer. Due to space restrictions, they were forced to meet in the library without proper board space. She believed that "not having a classroom [...] just opened everything up." She utilized Geometer's Sketchpad, and considered that to be the "beginning experience" where experimentation allowed structure to emerge. She described the class as follows:

We generally will have a day or two of experience with a concept on the computers. Then I'll assign them homework, read the chapter, do homework, without any class discussion and we don't come together and have class discussion until after all of that [...] And the kids are really getting into it. The groupwork at the round tables, they just come in and everything's out there, they just go to work [...] So having the experience, assigning the homework and then having the lecture has been good for the kids. They're actually talking to each other about math. A lot. [...] The conversation that's happening, the independence that's developing, is wonderful. It's really, really exciting.

She structured the course on principles of exploration and “experience first, either through Sketchpad or through an activity in class, drawing on that experience, [...] then coming together as a class to lecture about it”, and was excited by the results. She “made them really struggle and work at” getting the material. She said that the struggle for understanding made it more valuable for students: “the fact that they worked for two weeks [...] and couldn’t get the answers, didn’t come easily, it made it much more important to them.”

Thus, her summer RLE program experience impacted her design of one course, as well as her beliefs regarding the role of exploration and the relationships between school and research mathematics. Those conceptions are explored in more detail in the next section.

### **Deborah’s Conceptions of Mathematics Research**

In the previous section, I described how Deborah began to appreciate the importance of exploration in the history of mathematics. As she experienced learning through exploration prior to instruction for herself, she also appreciated it as a means by which student learning might proceed. Thus, she began to consider the history of mathematics as a model for the student learning process. As we discussed the nature of mathematics research, her shifting conceptions of it became clearer. She said her summer experience “reawakened that remembrance of what the heart of mathematics is, where it comes from. People work at these things a long time!”



During the second interview, when asked to respond to a hypothetical situation where a student asks her about mathematics research, she said she felt it was important to stress

the language of it. Working at the language of mathematics. That you think about math differently – it's not just doing problems and getting answers, it's mostly not getting answers. And working at [...] the not getting of answers. But it is digging deeper into your understanding of what's going on - structure, and writing it down. Conversations, [...] we attempt to have them talk about the math that we're doing, but that's nothing compared to what really happens at the higher levels.

Thus, she understood mathematics research in terms of struggling through exploration and “not getting answers”, something she experienced and identified as new and meaningful during her summer experience. Also, her emphasis on language, present at the outset of the summer as a result of her previous experiences as a teacher and a learner, was incorporated into her description of mathematics research. She incorporated aspects of her existing conceptions of student learning into her conceptions of mathematics research while also incorporating her experience with research into her conceptions of student learning. This fact indicates that she was constructing parallels between the two processes.

She confirmed that she saw similarities in the third interview. First, she described the geometry class discussed above, where experimentation and exploration led to the emergence of structure and understanding. She based her organization of the course on the RLE program and her understanding of mathematics research. Furthermore, when presented with the same hypothetical situation concerning the student asking about mathematics research, she

responded that such a discussion had begun to arise in her geometry course.

She said,

Several of them were like, “why didn’t I see that? That’s so obvious, why didn’t I see that?” And so we got into a discussion about a little bit about the history of mathematics, and how hundreds of years would go by [while ideas were developed], you know?

In response, she began to incorporate excerpts from a history of mathematics book in order to help students appreciate the time and struggle it took to develop mathematical ideas. As she said, “you expect it to happen overnight, but it doesn’t. I think they were, like, surprised or shocked into thinking that.” She felt like they understood and appreciated the nature of mathematical work “because they went through that struggle and it wasn’t obvious.” Thus, she believed her students gained understanding of mathematics research because their learning followed the same trajectory – the two processes were similar.

Furthermore, Deborah responded to the statement *when mathematicians do mathematics, they’re doing something fundamentally different than when students do mathematics* by saying:

It’s fundamentally the same. It’s just a different level [...] because I think they’re learners, also. I think they’re learning, research mathematicians are learning. And discovering, making observations, defining patterns, testing them. They don’t have an answer book (laughs) – it’s a little harder for them [...] they have to use their own minds [...] Students have someone directing them. That’s a fundamental difference between them, and students are studying the past, not the future.

So, though she recognized that the two processes had important differences, she believed them to be principally the same. For Deborah, both involved discovery, observation, pattern-sniffing, experimentation. Thus, over the course of the

summer program, Deborah reconsidered her beliefs about mathematics research, and came to see the process of it as similar to the process of student learning.

### **Discussion and Conclusion**

Deborah began the RLE program with mature conceptions about teaching, student learning, and herself as a teacher and learner. She had formulated her conceptions over the course of many years of teaching and some professional development experiences that were extremely influential. Because her conceptions were mature and robust, she did not see them as easily renegotiated. However, her experience had also fostered an attitude of self-reflection and an appreciation for “meaningful experiences”, so she was ready to evaluate her experience and incorporate aspects of it into her classroom as appropriate.

Because of her well-developed conceptual schemes, her existing conceptions influenced the way she responded to and interpreted her own summer experience. Though she had trouble finding a consistent group to work with, she believed it important to work with others, and mentioned “doing math with other people” as one of the most important parts of her experience. She also was more comfortable with uncertainty than many of the other participants, and accepted being unable to complete problems immediately. She believed that language, rather than grasping the concepts, was the most difficult aspect of mathematical learning. For her own experience, she was most sensitive to anything that was the “language of mathematics”. She also structured her

lessons so that students could make choices, and this emerged as a theme from her descriptions of her experience, as well.

Furthermore, her beliefs about the ways in which student learning occurs did not change significantly, nor did her expectations of what students were capable of independently from the teacher. In fact, her primary belief that student learning proceeds from the concrete to the abstract was reinforced by her time in the RLE program. However, Deborah's beliefs were not static. Because of her attention to student learning, she consistently looked for ways her experience related to her students and her classroom. And, it seemed that she found some principles that she felt would prove useful to her teaching. First, having problem sets that encouraged investigating material that had not been explicitly taught was frustrating, but empowering for her. The idea that long-term exploration prior to being taught the underlying structure might actually help one to see and better understand that structure was important to her experience. Her hypothetical learning trajectories changed to include more exploration at the beginning of the lesson, and she structured her new geometry course on that principle. Her summer experience also "reawakened" her knowledge about the nature of mathematics, which she saw as built upon exploration in order to see structure. Because exploration was meaningful in her own experience, and also was important to mathematics research, she constructed parallels between the research and learning processes, believing both to be the development of language and structure from concrete representations, manipulations, and explorations. She began to incorporate vignettes from the history of mathematics

into her geometry class, and considered how her classes might model the research process. She, more than any other interview participant, drew and made use of parallels between research and student learning.

Deborah, like the other interview participants, renegotiated her beliefs and attitudes toward student learning by means of incorporating her own learning into her existing belief systems. That is, she drew parallels between her own experience and that of her students. More than some of the others, she constructed parallels between research and learning, but these were nevertheless facilitated through her own experience, which mirrored both processes. Regardless, Deborah's pre-existing attitude of reflection and introspection led her to be open to changing and reconsidering her conceptions. She felt that her summer experience was meaningful and had an important impact on her conceptions and her practice.

### **Concluding Remarks**

The sections above offer a great deal of detail regarding the background, summer experience, and conceptions of the five teachers. The conceptual lens through which I view this work recognizes that each individual's experience is unique and idiosyncratic, and the descriptions above have attempted to treat them as such. At the same time, the conceptual framework acknowledges that individual experience takes place in a social and cultural setting, and that different individuals can experience similar phenomena as a result. Additionally, individuals may respond to a common stimulus in similar ways. In order to discuss the general impact of the RLE mathematics immersion program in

question, I turn now to describing the common threads that ran through these cases.

## CHAPTER V

### DISCUSSION AND CONCLUSIONS

#### **Introduction**

The case studies presented in the previous chapter provide insight into the experiences of each individual as they participated in the RLE program. The purpose of this chapter is to consider the impact of this particular mathematics research experience on teachers' conceptions of student learning more generally. Thus, I will first discuss the results of the belief survey, which was administered to the larger group of teachers. Though reliability issues prevented any hard conclusions from being drawn from the surveys, they did nevertheless indicate some possible areas where teacher conceptions may have been impacted. The details and possible explanations for those changes will be discussed in this chapter. Furthermore, I will highlight the issues that arose with the instrument itself, emphasizing their impact on the data and the development of the final interviews. I will then compare the case studies to each other in order to highlight common themes that run through them, including the changes in teacher conceptions that were observed as well as the roles of individual experience and changing conceptions of mathematics research in instigating those changes. Taking that information together with the belief survey data will create a fuller picture of the impact of the experience on the participants.

Connections to existing research will help to highlight the implications of the present study and its contributions to the field.

### **Belief Survey Data**

#### **Participant Data**

The belief survey was first administered during the first week of the RLE summer program during a weekly lunch meeting. Due to time constraints, the survey was passed out during the meeting and participants returned it over the course of that day and the next. Twenty-nine participants returned the surveys. The follow-up survey was administered in a similar manner during the final week of the program and, again twenty-nine participants returned the survey. Of this number, twenty-four returned both surveys. The statistics below were calculated for those twenty-four participants and provide background information and context for the rest of this study.

The twenty-four survey participants included eight second-summer teachers and sixteen first-summer teachers. The teachers had between zero and 25 years of teaching experience<sup>26</sup>, with a mean of 7.23 years and a median of 5.5 years. The upper and lower quartiles of the data were at 10 years and 3 years, respectively. They had been in their current position between zero and 13 years, with a mean of 3.71 years and a median of 3 years. The upper quartile fell at 5.5 years of experience, and ninety percent of the teachers had been in their current position less than 9 years. Thus, the population demonstrated a wide

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<sup>26</sup> Teachers with zero years of experience were those pre-service teachers completing the RLE program as part of a pre-service Master's degree.



range of teaching experience, but consisted mostly of teachers who had been in the profession for less than ten years. The “typical” teacher in the program had taught long enough to base her conceptions of student learning on interactions with students and to have a well-developed self-concept of her own teaching. However, she was new enough to still be considering those conceptions and did not yet consider herself an expert. The teachers’ participation in a summer professional development program is indicative of some openness to new ideas. Below, I will detail the responses of these participants to the survey and discuss the implications of those results regarding teacher beliefs and the impact of participation in the RLE program. I will refer to the first administration as the pre-test and the second as the post-test.

### **Category Reliability and Responses**

As described in the methodology chapter, the items on the belief survey were divided into three categories. Category I measured the teachers’ beliefs regarding the nature of student learning – the degree to which students constructed knowledge for themselves (agreement with that idea would have been indicated by a higher response score) or had it transmitted to them by an expert (indicated by a lower response score). Category II measured the degree to which teachers’ believed the processes of mathematics research and student mathematical learning in high school to be similar. Equivalently, this could be stated as the degree to which they believed mathematicians and students to be engaged in similar work. Higher response scores indicated a belief that the two were similar, while lower scores indicated a belief that they were dissimilar.

Category III measured the teachers' beliefs regarding the capabilities of their students to generate insights without significant teacher support. Higher scores indicated belief in student capability independent of the instructor. In order to assess the reliability of the set of items in each category for testing the same construct, Cronbach's alpha was calculated for the ten items in each category for both the pre-test and the post-test. The statistics, shown below, were consistently below the desired range of 0.7 and above.

TABLE 7: *Cronbach's alpha for Pre-test and Post-Test, Original Categories*

Category I		Category II		Category III	
Pre-Test	Post-Test	Pre-Test	Post-Test	Pre-Test	Post-Test
0.4891	0.5887	0.3619	0.6263	0.5193	0.5043

The reasons behind the wide disparity seen in category II is unclear. However, these results made it clear that specific items were adversely affecting the reliability of each category, so these items were removed. Specifically, item 28 was removed from category I, items 6 and 9 were removed from category II, and items 20 and 10 were removed from category III. Thus, the final results considered 9 items in category I, 8 items in category II, and 8 items in category III. The resulting sets of items are referred to as adjusted category I, adjusted category II, and adjusted category III. This change resulted in improved reliability scores, but they were still below the accepted 0.7 threshold. Removing further items from the categories resulted in minimal improvements to the alpha values.

TABLE 8: *Cronbach's alpha for Pre-test and Post-test, Adjusted Categories*

Adjusted Category I (item 28 removed)		Adjusted Category II (items 6 and 9 removed)		Adjusted Category III (items 20 and 10 removed)	
Pre-Test	Post-Test	Pre-Test	Post-Test	Pre-Test	Post-Test
0.5659	0.6375	0.5945	0.5519	0.6273	0.5863

Interestingly, removing item 6 from category II markedly increased alpha for the pre-test (from 0.3619 to 0.5538), but decreased it for the post-test (from 0.6263 to 0.5381). The reasons for this are somewhat unclear. However, it highlights the idea indicated by the alpha values themselves: that the belief survey categories were not reliably testing the same construct. I chose to remove items 6 and 9 as a way to maximize the alpha values for both the pretest and the posttest.

Thus, the belief survey was, at best, problematic, and hard conclusions could not be drawn from the mean response scores for the categories. However, it was nevertheless interesting and informative to consider the changes in the categorical mean response scores from the pretest to the posttest. Doing so provided directions and insights that were potentially useful during the more in-depth one-on-one interviews. Keeping in mind that the low reliability scores made it impossible to draw any significant conclusions from the data, the *p*-value yielded by the two-sample *t*-test using the mean response scores for category II rejected the null hypothesis that the means were the same. The null hypothesis was not rejected for categories I and III:

TABLE 9: *Data for Belief Survey Categories*

Adjusted Category	Category Mean Response Score – Pretest Mean	Category Mean Response Score – Posttest Mean	<i>p</i> -value (Null Hypothesis = means are equal)
I	3.6528	3.6157	0.2477
II	3.6616	3.7917	<b><i>0.0457</i></b>
III	3.5990	3.5708	0.2206

*Note:* *P*-values rejecting the null hypothesis for  $\alpha=0.05$  are shown in bold and italics.

Were the categories reliable, these statistics would indicate that the participants believed students construct their own knowledge rather than receiving it, that mathematics research and student learning are related, and that students are capable of independent breakthroughs. Furthermore, they would indicate that these beliefs were not strongly held. Such results would be consistent with the beliefs professed by the interview subjects. In general, the interview subjects weakly professed to believe that students constructed their own knowledge, that mathematics research and student learning were at least somewhat related, and that students were capable of independent insights up to a point. Though Deborah and Scott seemed to hold the belief that students construct their own knowledge more strongly than the others, in general, all three categories of beliefs were held by the interview subjects not as knowledge that significantly impacted their practice, but more as general assumptions regarding the nature of learning.

If the categories were reliable, the statistics shown above would indicate that only their category II beliefs, concerning the relationship between mathematics research and student learning, changed – that the group believed the two processes to be more closely related at the end of the program than they did at the beginning. Though the low reliability limits the strength of the conclusions that can be drawn from the data, these results were compelling enough to provide a direction to investigate during the final set of interviews. Indeed, it became clear from the interviews that the interview subjects were at least considering the relationships between mathematics research and student

learning more than they had in the past, something discussed in more detail below. Furthermore, previous investigations into mathematics immersion programs for pre-service or in-service teachers have indicated that such experiences have caused teachers to reconsider their beliefs about the nature of mathematics (Badertscher, 2007; McCrone, et al., 2008). Those previous results further underscore the importance of considering the interview subjects' beliefs about mathematics research and provide evidence that changes in teachers' category II beliefs would not be unprecedented.

Examining the belief survey categories yielded no significant conclusions due to the low inter-category reliability. However, a shift in the mean response scores of category II indicated that participants may have been reevaluating their beliefs regarding the relationships between research mathematics and student mathematical learning. In order to mine the belief survey for as much information as possible, I also considered teacher responses to individual items in order to determine any changes in the mean response score for each one. Any patterns that emerged from those items that showed significant change in mean response score could provide information regarding the impact of the experience on teachers' beliefs. Indeed, the mean response scores for a few items did change significantly from the pre-test to the post-test.

### **Individual Item Response Scores**

The mean response scores for each item on both the pre-test and the post-test are shown in Table 10. The mean response score was calculated for each item for each administration and two-sample Student's *t*-test used to determine the

significance of any change in the mean response score. The one-tailed  $p$ -values are reported in the table below. Those values that rejected the null hypothesis that the means are equal using a 0.05 level of significance are shown in bold italics. Those that rejected the null hypothesis using a 0.07 level are shown in italics. Using  $\alpha=0.07$  as the threshold for rejecting the null hypothesis means the test would erroneously reject a true null hypothesis 7% of the time. Though this is less rigorous than the traditional threshold of 0.05, it is still low enough to be of some interest, particularly because the results were principally used to suggest directions for further investigation rather than to draw hard conclusions. Furthermore, there was a noticeable gap between the eight  $p$ -values less than 0.07 (all of which were less than 0.065) and the others, all of which were greater than 0.1.

The  $p$ -values for items 1, 4, 8, 22, and 27 were all less than 0.05, indicating that the changes from the pre-test to the post-test were unlikely to be the result of random error. These items were in categories II, III, I, I, and II, respectively. In addition, the  $p$ -values for items 5, 9, and 18 were less than 0.065, greater than the 0.05 threshold, but still low enough to be of some interest. These items were from categories III, II, and III, respectively. These eight items consisted of two from category I and three from each of categories II and III.

TABLE 10 – Data for Individual Beliefs Survey Items

Item	Category	Positively (P) or Negatively (N) Stated	Pre-Test Mean Response Score	Post-Test Mean Response Score	p-value (Null Hypothesis = means are equal)
1	II	N	3.4167	3.8333	<b>0.0237</b>
2	I	P	3.1667	2.9167	0.1037
3	II	N	3.625	3.75	0.1885
4	III	P	3.458	3.9091	<b>0.0044</b>
5	III	N	4.2917	4.0	0.0552
6	II	P	2.5417	2.9091	0.1332
7	I,III	N	4.0417	3.9091	0.3142
8	I	P	3.1667	3.6363	<b>0.0023</b>
9	II	P	2.7083	3.0455	0.0647
10	I,III	P	4.5833	4.5909	0.50
11	I	N	3.625	3.8333	0.1638
12	I	P	3.5	3.5417	0.4071
13	III	N	2.2174	2.3333	0.3735
14	III	N	3.1304	3.1667	0.50
15	I	P	4.1667	4.2083	0.3570
16	II	P	4.0833	4.0417	0.4012
17	II	N	3.25	3.2083	0.4200
18	III	N	3.7083	3.375	0.0593
19	II	P	4.0	4.125	0.1885
20	III	P	3.624	3.4833	0.3850
21	II	P	3.875	4.0	0.1639
22	I	N	3.1667	2.75	<b>0.0284</b>
23	II	N	3.9167	3.8333	0.3236
24	III	P	4.125	4.2083	0.2129
25	I	N	3.4583	3.3333	0.1639
26	III	P	3.7083	3.75	0.4120
27	II	N	3.125	3.5417	<b>0.0191</b>
28	I	N	2.50	2.50	0.50

Note: *t*-scores indicating significant change for  $\alpha=0.05$  are shown in bold and italics. Those indicating significant change for  $\alpha=0.07$  are shown in italics.

The eight items highlighted above were distributed across all three categories, meaning no one theme emerged as dominant. However, consider the wording of the actual items:

**Item 1:** It is unrealistic to model a mathematics classroom on the behaviors of mathematicians. [Category II, *p*-value=0.0237]

**Item 4:** Students extend their current knowledge to solve types of problems they've never encountered before. [Category III,  $p$ -value=0.0044]

**Item 5:** Students shouldn't be asked to solve problems if they haven't already seen the procedures for doing so. [Category III,  $p$ -value=0.0552]

**Item 8:** Students can often figure out relationships between mathematical topics without being shown. [Category I,  $p$ -value=0.0023]

**Item 9:** Learning mathematics and researching mathematics involve the same amount of ambiguity. [Category II,  $p$ -value=0.0647]

**Item 18:** A teacher shouldn't test or quiz students with problems the students haven't already seen (other than perhaps changing the numbers). [Category III,  $p$ -value=0.0593]

**Item 22:** In order to be a good problem solver, it is important for students to follow directions. [Category I,  $p$ -value=0.0284]

**Item 27:** The thought processes involved in learning high school mathematics and researching mathematics are different. [Category II,  $p$ -value=0.0191]

Because of the inter-related nature of the concepts being tested, some items tended to overlap categories. Thus, for example, some items in category I were closely related to category III. Interestingly, this was the case for both of the items from category I listed above (both adapted from Fennema, Carpenter, & Loef, 1990). Thus, even though the eight items were distributed across all three categories, five of the eight (items 4,5,8,18, and 22) involved beliefs about student capabilities. Thus, the data provided by the belief survey indicated that the teachers were reconsidering their beliefs regarding student capabilities.



Of the five items, three of the mean response scores decreased (for items 5, 18, and 22), while two increased (for item 4 and 8). This, along with the low reliability for the categories, might explain why the mean response score category III as a whole did not change.

The individual item responses indicate that the experience may have led teachers to consider their expectations of student capability, though some items indicate those expectations increased while others indicated they decreased. It is unclear exactly why this happened, but there are a couple of possible contributing factors. First, the low reliability of the belief survey itself could be a contributing factor, and at the very least limits the conclusions that may be drawn. Secondly, recall that the program participants sat in the lecture with gifted high school students. These students often contributed to the discussion during lecture, and many of them held extremely advanced knowledge and critical thinking skills. Teachers' responses to these students varied during our informal discussions. They were universally impressed with the students, but some teachers began to consider that their own students might be capable of more than was being asked of them. Others, on the other hand, reacted in essentially the opposite direction – by determining that some students are extremely capable, but that their students were not like those participating in the RLE program and require much more support. One or both of these factors may have played a role in the observed changes.

Only limited conclusions may be drawn from the belief survey. At best, it indicated that teachers may have reconsidered their category II beliefs, believing

mathematics research and student learning to be more closely related after the RLE program than they did before, and their category III beliefs, reconsidering their beliefs regarding student capabilities, though not in a consistent direction. However, the low reliability of the categories made hard conclusions based on the data impossible. Nevertheless, these insights were valuable to shaping the focus of the final interviews with the interview subjects. The conclusions drawn from those interviews are discussed below.

More generally, I was left with reservations regarding the utility of belief surveys as a data-collection tool. They cannot offer fine-grained detail or nuances of a teachers' belief system, but are useful for large-scale data collection. Furthermore, even the most rigorously constructed surveys leave questions regarding interpretation that may impact the results (Philipp, 2007). In the case of this study, the belief survey was most useful as a data source for background information on the case study subjects. The remainder of this chapter will focus on the common themes that emerged from the case studies, incorporating those themes with the data from the belief survey in order to draw conclusions about the impact of the RLE program on teachers' conceptions of student learning.

## **Conclusions From Case Studies**

### **Introduction**

Chapter IV presented the five case studies developed from the task-based interviews. Themes emerged from each individual concerning their experience in the RLE program – the aspects of the program that they found most meaningful,

and the beliefs that they reconsidered or shifted as a result. Taking a coarser-grained look at the individual themes allowed me to consider more global themes that ran across all of the case studies. In particular, it allowed me to consider each of the research questions for the group rather than for each individual and to draw conclusions about the impact of the RLE program. That is, all of the individual interview participants internalized and made sense of their experiences in different, individualized ways (Cobb & Yackel, 1996). However, certain similarities emerged from their individual experiences. This section will describe and discuss those conclusions in detail. I will highlight how these results relate to and extend previous research results and connect them to information gathered from the belief survey in order to develop a global perspective on the project results.

### **Common Changes – The Role of Exploration and Empathy Toward**

#### **Students**

The case studies highlighted some common themes in the ways teacher conceptions changed. First, teachers all claimed that utilizing exploration as the first step in mathematical learning was impactful. They variously described that exploration as “data-collection”, “experimenting”, or “exploring”, and it consisted of working through multiple examples in order to notice patterns and make conjectures. Proving, extending and applying the concepts developed through the initial exploration came after. Each of the interview subjects discussed the value of looking for patterns and commonalities in multiple examples, a mathematical habit of mind that Cuoco, Goldenberg, and Mark (1996) called

*pattern-sniffing*<sup>27</sup>. Though it was not a new idea to every teacher (in particular, Scott, Joyce, and Deborah all incorporated something similar to some extent in their initial lessons), the amount of exploration included in the RLE program and the way understanding was expected to emerge from it (and in these case studies, *did* emerge for the teachers) was a new idea. Every teacher identified it as a key aspect of their own work in the program and as a key aspect of mathematics research. Furthermore, it was incorporated to various degrees into teachers' lesson plans as they came to believe it to be important for learning. For instance, Deborah incorporated some exploration into her algebra courses even before the program, but she subsequently remodeled her geometry class around "exploration first" principles, and was very excited by the results. Other teachers adjusted their hypothetical lessons to include more exploration or independent student work at the beginning. Even, Emily, whose conceptions of student learning, more than those of any other teacher, were focused on the teacher directing and guiding student learning, exhibited new attention to describing patterns and general ways of approaching problems during the observed classroom lesson. As teachers experienced learning for themselves through exploration<sup>28</sup>, they developed an appreciation for it as useful for learning.

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<sup>27</sup> One of several habits of mind they advocated as useful for student learning in mathematics.

<sup>28</sup> Recall that, during the summer, they were encouraged to "explore" by first attempting the numerical problems. This pattern was repeated during the second summer research project.

Secondly, there was a common shift in the teachers' perspectives on their students' experiences. It seems that the experience of being a student oneself led these teachers to a greater understanding of and empathy for their students. Scott's case is a good example. He came to the program with the expressed goal of experiencing learning in a context similar to that which he desired to provide for his students. The hypothetical learning trajectories that he constructed did not change, but the way he talked about student participation in the lesson became much more attentive to their feelings and experience rather than just their actions. That is, his discussions indicated that his participation, and in particular, feeling the confusion, frustration, and insecurity of a learner, increased his empathy for his students. Jennifer, Joyce, and Emily indicated a similar shift, while Deborah already exhibited a well-developed sense of her students' experience – a sense that was amplified as she reflected on her own experience. Previous research has suggested that professional development experiences that encourage teachers to reflect on student thinking and their work in the classroom can impact teachers' practice (Carpenter, et al., 1989; Sowder, 2007) and that professional development standards requiring deep coverage of content and attention to student learning could help professional development experiences have a greater impact on the participants (Hill, 2004). In the case of the RLE program, it seems that learning unfamiliar content prompted these teachers to independently reflect on and reconsider their views regarding their students' experience. In fact, they actively drew parallels between their own experience and that which they expected of their students, which I will outline

below. This occurred on an individual basis as the participants interpreted their experiences within the framework of their existing conceptions. Indeed, core conceptions remained consistent, and existing conceptions impacted the ways teachers interpreted their experiences.

### **Consistency and Impact of Primary Beliefs**

Overall, despite the changes described above, teachers' conceptions as revealed through their lesson trajectories remained mostly consistent. Each teacher maintained the same basic hypothetical learning trajectories throughout all three interviews. Some teachers made small changes, and the emphases of some of their lessons shifted, but, overall, there was a marked consistency in the conceptions guiding the interview subjects' lesson development. Aspects of each individual's conceptions changed over the course of the summer and their return to the classroom. For instance, Scott and Jennifer both became increasingly empathetic toward their students' experience, and Joyce began to place a greater emphasis on group work and exploration as important means by which learning proceeds. Deborah's belief in the power of exploration and doubt as part of the learning process was strengthened, as well. However, none of their primary beliefs, the ones that are most important for guiding one's practice, changed much, if at all. Perhaps in part because their beliefs were consistent with the organizing principles of the RLE summer program, the basic trajectories by which those four teachers believed student learning to proceed were unaffected. Emily was a particularly illustrative case because her conceptions were in many ways inconsistent with the philosophy of the RLE program.

Despite that inconsistency, her conceptions that knowledge developed through and consisted of procedural mastery and adherence to a teacher-centered classroom were too deeply ingrained to be impacted. In the case studies, I referred to those aspects of teacher beliefs that underwent change as teachers' perspective on student learning or as their conceptions regarding the means by which student learning proceeds. In other words, only peripheral beliefs changed, not those that were central or primary for individuals (see Thompson, 1992).

The interviews revealed a variety of primary beliefs for the interview subjects. Emily's goal of procedural fluency and her adherence to a teacher-centered classroom were primary beliefs – and they therefore influenced her interpretation of her experience more than they were influenced by it. Similarly, Deborah believed that student learning was an individualized process, and that a teacher's willingness to reflect on her own practice is vital to successful teaching. Joyce also believed in individualized learning trajectories and that group work was necessary for effective learning. Scott believed in guided, structured problem-solving as the key means by which learning proceeds. Jennifer's primary beliefs were more difficult to identify, but also included structured, sequential problem-solving. None of these conceptions were impacted significantly by their participation in the RLE summer program. Note, however, that just because a conception was not impacted by the experience does not necessarily mean it was a primary belief for that individual. For example, Joyce's beliefs regarding student motivation and interest were unaffected over the course

of the summer. However, their interest level did not drive her practice to any great extent. Rather, she saw student interest as an aspect of her classroom that fit into her larger model of how students learn – her conceptions regarding student interest were not core. The converse, however, did seem to be true – that primary beliefs regarding student learning were not significantly impacted by the experience, which was consistent with research indicating that primary beliefs are those that are most influential on an individual's action and are extremely resilient and resistant to change (Philipp, 2007).

In fact, primary beliefs impacted the ways teachers interpreted their experience in the RLE program in general, and thus shaped the ways in which they came to understand their summer experience. One's conceptions impact the ways in which one encounters and interprets the world (Thompson, 1992; Philipp, 2007), and this was illustrated by the teachers in my study. For instance, Emily, believing strongly in teacher-directed instruction, approached the problem sets from the perspective of trying to determine what their greater purpose was. That is, she felt as though her role as a learner was to determine what the organizers (teachers) were trying to have her learn. Moreover, her predisposition to consider procedural fluency as the definition of mastery led her to define her own learning goals in terms of procedural mastery and adopt it as the criteria by which her learning should be judged. Deborah's conception that students react individually and need time to explore led her to respond to the same situation differently – taking things at her own pace and allowing herself to gain understanding as it developed. She also defined success differently for herself,



wanting to understand “the concept”, not just to be able to complete problems. Her predisposition toward considering the language of mathematics and decision-making as part of mathematical work led her to highlight those aspects of mathematics research. For Emily, on the other hand, mathematics research was much more about action than it was for Deborah. These two cases (along with others - Scott, for instance, in considering himself as a student, was predisposed to think of the experience in terms of his school’s “student as doer, teacher as coach” philosophy) demonstrated that the preexisting conceptions of individuals are likely to impact their experience, and play a significant role in determining what they get out of it. Teachers’ conceptions’ played a role in shaping their experience, and thus influenced the way their peripheral conceptions were ultimately impacted.

In many ways, the changes to interview participants’ conceptions of student learning were a question of degree – a shift in the empathy displayed for students’ experience, in the amount of exploration incorporated into lessons, in the responsibility placed on the group rather than the individual for knowledge construction, in the willingness to allow students to struggle. Their experience impacted their conceptions as indicated above, but they were more inclined to understand their experience in light of their existing conceptions and alter, rather than replace, those conceptions in order to integrate their new experience. Because teachers interpreted their learning in the context of their conceptions, they were able to integrate or dismiss any contradictions to them. Thus, these case studies indicate that teachers’ existing conceptions impacted the way they

made sense of this mathematics immersion experience, but that peripheral aspects of those conceptions also shifted in response to that experience. They adjusted their conceptions of how students learn in a way that accommodated and explained their own learning experiences.

### **Teacher Attribution of Learning**

Heinz, et al. (2000) noted that, when teachers come to an understanding through a particular activity, they often attribute that understanding as a property of the activity itself. That is, teachers don't always consider the cognition that takes place as they learn through a particular activity, but instead see the knowledge growth to be a result of the activity itself. Thus, they believe activities that they have found beneficial for their own knowledge growth will also be beneficial for their students' knowledge growth. This study indicates that such attribution did occur for some of the interview subjects. Joyce, for instance, believed exploration and group work were important avenues for student learning, but she seemed unsure as to how exactly student gained knowledge through them. That is, she assumed that students would learn content by virtue of doing those things without any attention to *how* the knowledge development occurred. Such assumptions were a recurring theme among the teachers – as the participants made sense of their experience, the tendency was to reflect on *what* they felt led to their learning rather than *how* that learning occurred. As a result, the changes to their conceptions reflected that. No teacher showed any changes in their conception of the cognitive actions or ways of thinking that they expected their students would undertake as they learned. With the exception of

Deborah, the teachers paid little to no attention to any of the reasoning processes and mental actions necessary for knowledge to progress through mathematical activities. Rather, the implication of their discussions was often that knowledge would develop by virtue of participating in certain activities or taking certain actions. In other words, learning was viewed as a product of the activities rather than as a product of cognition that those activities might prompt. However, the fact that they saw activities that were meaningful to them as likely to be meaningful for students highlights a significant theme that emerged from the interviews: Changes in teacher conceptions occurred largely due to participants drawing parallels between their own learning experience and that which they expected of their students. However, the focus of those parallels was on the actions that produced learning without reflecting substantial consideration or understanding of the psychological processes underlying it.

### **Parallels Between a Teacher's Own Experiences and That of Their Students**

The fact that the teachers felt that they better understood their students as a result of becoming students themselves is indicative of the most impactful part of the RLE program – placing teachers in powerful and unfamiliar learning environments. Indeed, in every case, teachers drew parallels between their own experiences in the program and that of their students. The changes to their conceptions of student learning and/or their perspectives on student learning that did occur mirrored the aspects of their summer learning experience that were most meaningful to them. For example, Joyce felt that her partnership with Chad and the “data collection” and “experimentation” phase of her project were

extremely important to her knowledge development. Correspondingly, her hypothetical learning trajectories began to emphasize the role of the group in moving individual learning forward and to incorporate exploration at the beginning of the lesson. Note that these changes were in accordance with her primary beliefs. However, peripheral beliefs concerning the way group members contribute to knowledge construction and the learning that can occur through long-term “data-collection” were impacted in accordance with her own learning experience.

Scott’s conceptions of student learning changed as he became more empathetic toward his students’ experience of feeling unfamiliar. Again, his primary beliefs and the hypothetical learning trajectories he developed in the interviews were unaffected, but his perspective on student learning changed. Jennifer’s conceptions of student learning stayed mostly consistent, but her experience led her to emphasize questioning and appreciate the power of coming to conclusions for oneself. Her lessons incorporated more discussion about connections and justifications. All of these were significant parts of her summer learning experience. While Deborah did not believe that the struggle to understand was a new experience for her, she did say that it “reminded” her that mathematics was and is built through significant work, conjecture, and experimentation over time. That prompted her to shift her conceptions of how students in her geometry class might effectively learn the material. Though Emily’s conceptions did not change significantly, the changes that were observed – such as her new appeal to her students’ “instinct” when teaching and some

unconscious attention to “connections” between problems – were rooted in her own learning experience during the summer RLE program. Thus, the changes observed in each teacher mirrored their own experience - these teachers were constructing parallels between their own learning and that which they expected of their students. School background and a teacher’s own learning experiences are a major source of beliefs in general and conceptions about student learning in particular (Clark and Person, 1986; Thompson, 1992), and teachers will often explain or understand student difficulties in terms of their own knowledge. Simon’s *Teacher Learning Cycle* (1994) posited that teacher learning begins with the teacher’s own mathematical learning. The results of my study are consistent with those previous studies. The relation of this study to Simon’s work is specifically discussed below.

### **Implications for Simon’s Teacher Learning Cycle**

Simon’s (1994) *Teacher Learning Cycle*, described in Chapter II, posits that a teacher’s own mathematical learning plays a vital role in their development of all kinds of knowledge, including knowledge of mathematics, theories of mathematics learning, understanding students’ learning, instructional planning, and teaching. He described the *Mathematics Learning Cycle*, in which teachers explore mathematical situations, identify the concepts involved, and then apply them, which leads to further exploration of mathematical situations<sup>29</sup>. He described similar learning cycles for *teaching, developing knowledge of*

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<sup>29</sup> See Figure 1 in Chapter II (p. 64)

*mathematics, developing theories of mathematical learning, understanding students' learning, and instructional planning.* In each case, the cycle is similar to the *Mathematics Learning Cycle*, except that the “explore mathematical situations” phase is replaced by the mathematics learning cycle. That is, the situation that precedes concept identification (followed by application), and, thus, instigates learning, is mathematical learning. The case studies here would seem to support this for teachers. In these cases, personal mathematical learning experiences instigated knowledge development. Teachers' own mathematical learning was incorporated into their negotiation, construction, and alteration of knowledge and belief systems regarding the nature of mathematics, student learning, and instructional planning.

Furthermore, this study highlighted the nature of the “application” phase once concepts have been identified. As discussed above, the participants tended to attribute learning as a property of the classroom activities – such as “exploration” – without considering the cognition that underlay the learning they experienced through the activities. Thus, teachers identified practical, action-oriented concepts of students learning (*what* they should do rather than *how* learning occurs), and the application of those concepts consisted of considering how the learning activities would or would not translate to their classroom.

Thus, the case studies examined here lend support to Simon's idea that mathematics learning instigates the formation of conceptions regarding student learning. However, they also demonstrate that teachers engaging in the learning cycles he described may not be developing deep or well-rounded knowledge of

the subject. They are, however, developing *some* conceptions of the subject, but one should not assume that learning mathematics, even in a mathematics immersion setting such as the RLE program, will necessarily prompt teachers to develop conceptions outside of adjusting their expectations of students to be in line with their own experience.

### **Parallels Between Mathematics Research and Student Learning**

The belief survey data, while not reliable, indicated that participation in the summer RLE program may have led teachers to consider the relationships between mathematics research and student learning. The interviews confirmed that some, but certainly not all, teachers indeed drew parallels between mathematics research and student learning. Deborah seemed most powerfully affected by considering similarities between the two processes. She noted that her summer experience had “reawakened that remembrance of what the heart of mathematics is, where it comes from.” She saw mathematics research as a model for student learning, characterizing mathematicians as “learners.” The geometry course that she taught during the subsequent school year exhibited how she came to see her students as modelers of the mathematics research process. In that course, she intentionally allowed students to work on their own and struggle long past when it was comfortable to do so.

Other teachers saw similarities to various extents. Joyce, for instance, believed the two processes to be similar to the extent that both involved exploring data and multiple examples in order to see patterns and make generalizations. Jennifer saw them as similar, but believed students were not capable of the

same sorts of insights and were engaging in the work in a simplified manner. Scott noted some similarities between mathematics research and student learning, but his conceptions of mathematics research were incomplete and ill-formed. He admitted to being unsure what true “mathematical culture” would look like, but still believed there were some similarities between research and mathematics.

Scott's case highlights what seemed to be a general pattern regarding how teachers came to see research and learning as parallel. He did not develop a firm understanding of mathematics research, so the similarities that he saw were only developed through his own learning experience. That is, teachers were utilizing their own learning experience when considering their students' learning while, at the same time, their personal learning was taking place in a mathematics research-like environment. The fact that their experience was meant to mimic mathematics research was stressed to them by the organizers. Thus, since their own learning was perceived to be similar to mathematics research, and they drew parallels between their own learning and that of their students, they came to see mathematics learning and student learning as similar processes (by the transitive property, in some sense). The participants' understanding of mathematics research was based primarily on an “explore, experiment, and look for structure and generalities” model. This was the basic model discussed by the course organizers. Though the teachers' descriptions were true to mathematics research as described by researchers themselves, and, in particular, highlight the “messy” parts that Muir (1996) and others note are



often hidden, they lacked the nuanced understanding of the process conveyed by those with mathematics research experience<sup>30</sup>. Some participants believed they had participated in, if not genuine mathematics research, something very close (Joyce, for instance). Others took an attitude that they must have been simulating research because they were told so, but they didn't feel as though they themselves were truly behaving as mathematicians (Jennifer, for example).

McCrone, et al. (2008) and Badertscher (2007) both showed that mathematics immersion experience can impact teachers' beliefs about the nature of mathematics. The interview data indicated that the RLE experience did lead these teachers to reevaluate their beliefs regarding the nature of mathematics research, and even to consider parallels between it and student learning. However, the changes in their conceptions of mathematics did not appear to be responsible for the observed changes in their conceptions of student learning. I make this claim first because of the obvious parallels between teachers' own learning experience and that which they expected of their students. It was clear that experiencing learning for themselves was the primary factor in changing their conceptions. Secondly, however, the limited view of mathematics research espoused by the participants indicates that most simply took the organizers at their word regarding the nature of mathematics research – that it still did not hold great meaning to them personally. Thus, the parallels between student learning and mathematics research that some of them drew were facilitated by the fact

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<sup>30</sup> see the descriptions provided in Hadamard, 1945

that they were expecting students to learn in ways similar to themselves, and their learning was taking place in a context that was described to them as research-like.

### **Teacher Construction or Alteration of Conceptions Proceeds Through Experience**

This study took the conceptual viewpoint that students construct their own knowledge, but that social influences play a major role in that construction, acknowledging that the role of the individual and the role of the group and the context are often impossible to separate. A great deal of research supports the notion that students learn most effectively through experience and active participation, guided by an expert, rather than through transmission from an expert (cf., Lester, 2007). In this study, that principle held true with teachers' construction of conceptions about mathematics, student learning, and the relationships between the two. The most profoundly impactful aspect of the RLE program for them was the opportunity to experience learning from the perspective of a student. Furthermore, those aspects of each individual's experience that were most meaningful were reflected in the changes observed in their discussions of student learning. Thus, this study indicates that, just as content knowledge is more meaningful and highly-developed when students "discover" it in some sense, changes to conceptions are more likely to occur (that is, teachers are most likely to construct new conceptions or alter existing ones) and to be meaningful to teachers if they "discover" these changes for themselves - if they come to them as the result of experience that challenges their existing

conceptions enough to warrant change. However, those changes may only occur within the context of a teacher's existing conceptual scheme, with primary beliefs remaining unaltered.

## **Summary and Conclusion**

### **Summary**

The belief surveys were, in the end, unreliable, limiting the conclusions that could be drawn from the data. However, they did suggest that the RLE program could have led teachers to shift their conceptions regarding the relationships between school and research mathematics and regarding the capabilities of students. These were areas of particular interest when examining the interview-based case studies. The interviews were inconclusive regarding teachers' expectations of student capabilities. By incorporating more initial "exploration", the participants betrayed a belief that students can develop generalizations from that exploration, though most of them tempered that expectation, adding scaffolded questioning or teacher guidance to the activities. Overall, teachers' criteria for defining lessons as successful did not change significantly. The evidence from the belief survey for changing conceptions in that category were sketchy at best, and did not indicate change in any consistent direction. The interview participants did not exhibit any particular pattern of change in this area. However, the interviews revealed that the individual interview subjects did begin to see relationships between school and research mathematics, mediated by their own experience with mathematics learning in a research-like setting. Indeed, of all aspects of the RLE experience, it was their

own learning experience that was most significantly drawn upon when considering how students learn. However, that was not enough to prompt changes to teachers' primary beliefs. The changes that were observed, which included an increased attention to the role of exploration or data-collection for motivating concepts and increased empathy for students' feelings as they learn new material, were shifts in the degree to which peripheral conceptions were held rather than wholesale changes. That is, those shifts took place within the framework of teachers' existing belief structures. The primary beliefs therefore shaped how teachers interpreted their experience, and peripheral beliefs were shaped as their conception systems accommodated that experience.

### **Conclusion**

As mathematics immersion has become a popular professional development model, debate has arisen regarding the true effectiveness of such experiences, with some researchers encouraging the research community to refrain from assuming that these experiences are having the desired impact (Proulx & Bednarz, 2009). Debate has also arisen regarding the relationship between school and research mathematics (Mendick, 2008; Watson, 2008; Zazkis, 2008). This study contributes to that conversation in several ways. First, the case studies of teachers participating in just such a program offer detailed accounts of teachers' experiences. An analysis of the case studies provided insight into the impact that mathematics immersion has on in-service teachers' conceptions of student learning, and demonstrated that, while the program did not instigate changes to teachers' core conceptions, it did impact their peripheral

conceptions. Furthermore, the themes that emerged from that analysis demonstrated some of the ways teachers make sense of their own learning experiences when considering those of their students. The study also has implications for the research community's understanding of the role of belief structures in determining how teachers experience professional development and how those belief structures are altered to accommodate those experiences. These contributions, along with methodological contributions, limitations of the study and possible directions for future research, are discussed in detail in the next chapter.

## CHAPTER VI

### CONCLUSION AND DIRECTIONS FOR FUTURE RESEARCH

#### **Conclusions and Contributions to the Field**

##### **Introduction**

Recall Cuoco's (2001) provocative claim that "the best high school teachers are those who have a research-like experience in mathematics." No one study could confirm or deny that claim (in part because "best" is a rather inexact descriptor), but, as discussed earlier, a body of research has begun to emerge regarding the effectiveness of "mathematics immersion" experiences. Amongst that research, there has been some debate regarding the true effectiveness of such experiences (Proulx & Bednarz, 2009). Another area of debate that strikes at the motivation behind mathematics research experience concerns the relationship between school and research mathematics (Mendick, 2008; Watson, 2008; Zazkis, 2008). Research of mathematics immersion experiences can also help clarify that relationship.

This research project contributes to the body of knowledge regarding mathematics immersion experiences. In particular, it details how participants in one such program considered their own students' learning as they made sense of their experience, thus examining the impact of the program on teachers. Since the participants in this research-like program were school teachers, they

were uniquely situated in the intersection of school mathematics and mathematics research. The research therefore offers some information regarding the relationship between the two as seen by those who matter most for the purposes of student education – secondary educators.

In addition to providing information regarding the effectiveness of mathematics immersion programs, the study offers a fine-grained account of teachers negotiating conceptions of student learning. These conceptions are important for a teacher's work prior to delivering a lesson, such as how they construct their lessons and develop their classroom to be conducive to learning (Penso & Shoham, 2003). Thus, insight into the development of conceptions of student learning is significant for understanding teacher knowledge and practice. In particular, the manner in which teachers' conception schemes both affected and were affected by the experience demonstrated the complexity and the simultaneous rigidity and fluidity of teacher conceptions. The fact that conceptions were affected most powerfully through experience demonstrates the importance and influence of learning experiences in mathematics for shaping conceptions regarding teaching and learning.

Finally, the methodology itself contributes an adaptable way of accessing teacher beliefs. Task-based clinical interviews have long been used to investigate student knowledge and teacher understanding (Confrey, 1981). However, this study used clinical interviews centered around lesson planning tasks in order to understand teachers' conceptions. Such interviews offer a promising way of understanding those beliefs that are most important to the work

of the interview subjects. Below, all of these contributions are discussed in some detail. First, however, it is important to acknowledge the limitations of this particular study in order to set those results in the proper context.

### **Limitations of the Study**

The major limitation of this study is a lack of generalizability. It examined only one mathematics immersion program, and such programs have taken on a variety of different forms<sup>31</sup>. Furthermore, the belief survey, meant to be the large-scale data collection tool, lacked reliability, limiting the conclusions that could be drawn from the data for the entire group. Thus, the conclusions are drawn, for the most part, from the five case studies based on the series of interviews.

Though there has been criticism of the lack of generalizability of case studies, Flyvbjerg (2004) illustrated that case studies, even of atypical or extreme cases, can offer rich information about a particular phenomenon. The cases in this study were not selected to be representative or to illustrate extremes, but the variation across initial conceptions and experience yielded enough information to valuably describe the situation in question – teachers participating in a mathematics immersion experience and considering their conceptions of student learning. Also, by comparing and contrasting several cases, results become more generalizable (Van Wylsberghe & Khan, 2007). Nevertheless, the study does not provide large-scale, statistically-rigorous conclusions. Rather, the

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<sup>31</sup> For descriptions of various programs, see Chapter II and Badertscher, 2007; Chazan, et al., 2007; Cuoco, Goldenberg, & Mark, 1996; McCrone, et al., 2008; Stevens, et al., 2001



information is more localized, but also more nuanced and thus, in some ways, more informative.

The conclusions are also limited to the domain of teacher conceptions during and immediately following the immersion experience. As a result of the timetable of the study, I was unable to draw any conclusions regarding long-term changes to teachers' practices or impacts on student learning and achievement. The goal of any professional development program is to ultimately improve student knowledge and understanding through instruction. An assessment of such outcomes was outside the realm of this study. Previous studies that have addressed the relationships between teacher conceptions, classroom practices, and student achievement (see Phillip, 2007; Thompson, 1992) offer some possible outcomes based on the observed changes to teacher conceptions. In this study, the classroom observation and the third interview provided some insight into changes to teachers' practice during the semester immediately following the summer RLE program. However, without follow-up study, there is no way of knowing how teachers processed, interpreted, and were impacted by their RLE program experience over the long term.

The study was also unable to delineate differences between the impact of the first summer and that of the second summer, when teachers undertook a more in-depth research simulation. The number of participants in the belief survey was too small to be significant for answering those questions, and the data yielded by the survey proved problematic, anyway. By interviewing both first- and second-year RLE program participants, I was able to gain some insight

into the differences through the development of the case studies. However, the small numbers again made it difficult to draw conclusions about any differences between the two groups. No matter how many years they had been attending, the teachers' experiences exhibited a great many similarities. The nature of mathematics research (an "explore then conjecture and axiomatize" model) was emphasized with both groups, and both had some simulation of that process, though the second-summer research project was more overtly "research-like." Nevertheless, the study offers insight into how a mathematics immersion program, and, in particular, a mathematics research experience, impacts teachers' conceptions.

Finally, it should be acknowledged that interpretation of the interview data took place through the conceptual lens as discussed in Chapter II. Though that lens was chosen, in part, because of its usefulness for the questions at hand, it is possible that other lenses could highlight different aspects of the data and lead to different, though not conflicting, conclusions. Thus, the conclusions drawn here are likely only part of what might be gathered from this data. Nevertheless, I make the case below that they are significant conclusions.

### **Effectiveness of Mathematics Immersion**

This study supports previous claims that mathematics immersion (in particular, simulation of mathematics research) is an impactful and formative experience for teachers (cf., Badertscher, 2007; Chazan, et al., 2007; Marshall, 2008; McCrone, et al., 2008; Stevens, et al., 2001). In particular, it demonstrates that teachers' conceptions of student learning in mathematics are shaped by their

own experiences as learners. Previous research has shown a variety of ways in which mathematics immersion programs have impacted teachers – for instance, their conceptions of mathematics (Badertscher, 2007; McCrone, et al., 2008), excitement about mathematics (Chazan, et al., 2007; Marshall, 2008), and anecdotal accounts of their teaching (Stevens, et al., 2001). However, conceptions of student learning have not been considered before. The results of this study indicated that teachers engaged in mathematics research simulations do consider the learning processes of their students. Furthermore, for the teachers studied here, the experience of being a learner in an unfamiliar setting was the most profound aspect of the program for shifting their conceptions of student learning. Teachers gained an appreciation for extended exploration at the outset of learning a new topic and became more empathetic toward their students. Their conceptions also changed in idiosyncratic ways that corresponded with each individual's experience. They also began to see connections between student learning and mathematics research, but these were mediated by the teachers' experiences in a "research-like" setting.

Thus, this mathematics immersion program was effective in moving teachers' conception of student learning. However, it did not change their primary beliefs – those most important for determining their practice (Philipp, 2007) – even when those beliefs were in conflict with the goals of the program, as in Emily's case. Furthermore, teachers' primary beliefs played a significant role in determining how the teachers understood their experience. All changes took place within the context of their core conceptual schemes. The results

therefore indicate that even a very powerful (as described by the participants) and unfamiliar learning situation was not sufficient to change these teachers' primary beliefs. It should be noted, however, that does not mean that the primary beliefs of every teacher in the program were unaffected. Badertscher (2007) examined two teachers as they participated in a mathematics immersion program and found that their existing views of mathematics as a discipline were a significant determinant of the way they interpreted their experience and their own mathematical work. Furthermore, the teachers' conceptions of mathematics as a discipline were affected, but only the teacher who had found the rigidity of her previous experience problematic accepted the new conceptions. The other teacher, who liked mathematics principally because of the rigidity disliked the course and rejected the challenges it presented to her existing conceptions. Similarly, in this study, teachers took certain common ideas from the program (exploration during the investigation process and empathy for students), but otherwise took from the experience that which was most important to their own learning. One aspect of those changes was considering the relationship between school and research mathematics.

### **School and Research Mathematics**

The RLE program used the parallels between the mathematics research process and the student learning process<sup>32</sup> as a foundational philosophy. However, there is debate among mathematics researchers regarding the utility of

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<sup>32</sup> I won't discuss these in depth here, but they are outlined in detail in Chapter II.

modeling student learning activities and experiences on the mathematics research process, with some arguing that school mathematics is essentially its own discipline, separate from the work of mathematicians (Watson, 2008). Others argue that intersections do exist (Zazkis, 2008), and, where they don't, it does not mean that dismissing the goal of behaving like mathematicians is appropriate (Mendick, 2008). The teachers in this study found meaningful similarities between mathematics research and student learning. Deborah, in particular, saw the historical development of mathematics as a model for how her students could learn the subject. The others saw parallels mostly in the role exploration of multiple problems prior to making generalizations could play in both processes. Scott, while identifying such similarities, did not develop a robust conception regarding the nature of mathematics, so Deborah's response was not universal. In fact, given her inclination toward personal reflection and her background in mathematics and science, her case may have been exceptional not just among the interview subjects, but also among teachers in general. Indeed, the three other case study teachers fell between Scott and Deborah. Joyce saw the two processes as quite similar, drawing mostly on the exploration concept discussed earlier, while Jennifer and Emily saw some similarities, but believed students were taking part in a very simplified approximation of mathematics research and still required teacher intervention and guidance. Deborah, Joyce, and Emily had more extensive mathematics backgrounds than the others, though Emily seemed to view her mathematical work as procedurally-based. Thus, each participants' background may have

played a role in their willingness or ability to draw parallels between research and student learning.

McCrone, et al. (2008), working with pre-service teachers, showed that experience with mathematics research can impact teacher beliefs regarding the nature of mathematics. This study confirms that research-like experiences can have some impact on teacher beliefs about mathematics. However, these changes took shape according to those aspects of learning that were most important to the individual. The exploration theme was common because it was stressed to the participants. Furthermore, the experience did not, in most cases, lead directly to changes in their conceptions of student learning. Rather, the relationships that teachers saw between research and learning were the result of parallels being drawn between their own experience as learners in a research-like setting and their students' learning trajectories. Research and school mathematics only intersected through the teachers' personal learning experiences. Because this study did not examine student learning, it was not designed to determine whether mathematics research is an appropriate model for student learning. It does indicate, however, that mathematics research is a model for teacher learning that may impact teachers' peripheral beliefs regarding student learning in ways that have some effect on their teaching. This does not answer the question of precisely how related the two domains are, but it does indicate that school and research mathematics do not have to be treated as totally disparate, separate domains.

## **The Interaction of Experience and Conceptions**

Beliefs, values, and knowledge exist in complex, inter-related groups that mutually influence each other according to the quasi-logicism of the individual possessing them (Furinghetti & Pehkonene, 2002; Green, 1971; Thompson, 1992). Any investigation regarding the conceptions of individuals must therefore acknowledge and account for the inherent complexity of that exercise. In this case, the investigation into the individual teachers revealed that complexity and demonstrated how conceptual schemes both influence the way teacher experience a learning situation and are influenced by that experience. The five interview subjects all highlighted slightly different (though certainly related) themes from the summer RLE program. Each individual's background and existing beliefs led them to understand the experience differently. For instance, Emily's conceptions regarding the primacy of procedural fluency and teacher direction led her to view herself as a student in just such a setting. She defined her own success through her ability to do problems and considered seeking the desired, at times hidden, agenda of the organizers to be her role as a learner. Similarly, Scott came to the program in order to learn in a setting similar to his school's "student as doer, teacher as coach" philosophy (which summarized his personal conceptions of teaching and learning, as well), and he viewed his participation in the program through that lens. Partly because of those views, the hypothetical learning trajectories he constructed for his students did not change, but his perspective on their experience did – he considered the students thought processes and feelings as they learned new material in a way he had not at the

outset of the program. The existing conceptual schemes and, in particular, the primary beliefs of the teachers shaped that which they took from the RLE summer program.

At the same time, their peripheral beliefs shifted as their conceptions were adjusted in order to make sense of the learning they experienced. To use the two examples from above, Emily began to stress “instinct” and connections between multiple examples even as she stressed to students that solving multiple examples was the path to and definition of understanding. Scott’s perspective on the student experience was shifted. In all cases, teachers were able to mostly accommodate their experience in the RLE summer program within their existing conceptions. There was not sufficient discord to prompt them to alter their primary beliefs, though peripheral beliefs did change. In some cases, this was due to consistency between teacher conceptions and program philosophies. Emily’s conceptions were in many ways contradictory to the program’s philosophies, but, and this is the key idea, she did not see them as such. Her preconceptions allowed her to interpret her experience within her existing schemes, and changes only occurred within those. Though other teacher conceptions exhibited less tension with the program, the same was true.

Thus, professional development programs should consider teachers’ preconceptions at the outset of the program. This study indicates that, similar to students in the classroom (see Fuson, Kalchman, & Bransford, 2005), preconceptions play a significant role in how teachers interpret their experiences. Preconceptions are therefore significant factors in how the professional



development program impacts classroom practice and, ultimately, student learning. Teachers can often accommodate even seemingly contradictory experiences within their conceptual schemes, so if one of a program's goals is to impact teacher conceptions in a particular way, it may be necessary to focus teachers' attention on what their conceptions are and what they mean for student learning. Organizers should be aware that, even when it might seem like philosophies and goals are clear, that does not mean participants will interpret them in the manner that is expected or intended.

Furthermore, professional development programs should be designed with an awareness of what is reasonable and possible. As described by previous research (see Philipp, 2007; Thompson, 1992) and supported by this study, teacher conceptions, particularly primary beliefs and knowledge, are difficult to change. In this investigation, an intense, multi-week summer program shifted some peripheral beliefs and caused teachers to reconsider their conceptions of mathematics research and perspectives on student learning, but did not shift their primary beliefs. A summer program is likely insufficient in the face of years of experience as a student, when beliefs are often formed (Clark and Person, 1986; Thompson, 1992), unless preexisting conceptions are in line with the philosophies of the program.

### **Research Methodology**

In addition to contributing to the body of knowledge regarding mathematics immersion and conceptions of student learning, this study also illustrates the usefulness of the task-based interview for understanding teacher

conceptions. The task-based interview has long been used to gain information about student knowledge and understanding (Confrey, 1981). Using lesson planning tasks to target teacher conceptions of student learning is a new and adaptable method for researching teacher conceptions, which are difficult to ascertain with accuracy. However, carefully constructed interviews allowed me to observe teachers using and discussing those conceptions that were most meaningful for their practice, and to watch them make sense of their professional development experience within the context of classroom practice as they discussed that practice. The method could easily be adapted for research programs that focus on beliefs or knowledge, whether or not they are attached to a particular professional development model. Indeed, interviews such as the ones utilized in this study could prove useful in research programs that extend the knowledge gathered here. I now turn to discussing these possible future directions.

### **Directions for Future Research**

#### **Research Methodology**

As indicated above, the interview methodology presented here offers a flexible way of obtaining data on teachers' conceptions of student learning. Beliefs are difficult to measure because they must be inferred, and Likert scale surveys such as the one used in this study offer only limited insight, necessitating other data-gathering techniques (Ambrose, et al., 2003). By standardizing the interviews and refining the analysis to develop a common rubric, a researcher could utilize the lesson-planning task-based interviews on a much larger scale

and compare teachers across several cases to a common standard. On the other hand, a series of such interviews coupled with observations of teacher actually teaching their lessons could yield an extremely detailed, comprehensive picture of the conceptions impacting a teacher's practice. Thus, the interview structures utilized in this study could be extended in natural and useful ways for the exploration of teacher conceptions. A close examination of the strengths and weaknesses of the task-based interview and a deep treatment of the theoretical underpinnings<sup>33</sup> could also prove valuable to the research community as a whole.

The belief survey used in this project exhibited several problems. In general, Likert-scale belief surveys present some problems, particularly regarding uncertain interpretation and a lack of sensitivity to the strength with which individuals hold their beliefs (Ambrose, et al., 2003; McGuire, 1969, cited in Philipp, 2007; Philipp, 2007). Nevertheless, they have proven valuable for accessing beliefs, particularly on a large scale. With a larger sample and a more extensive piloting process, the belief survey used here could eventually prove useful for large-scale data collection regarding teacher beliefs about student learning. In particular, this study could be scaled up to analyze a number of different mathematics immersion programs (or other professional development programs) in order to understand the benefits of the model and compare different programs. In such a case, a survey instrument could be a significant asset.

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<sup>33</sup> Such as that which was provided by Confrey (1981) for task-based interviews with students

### **Mathematics Immersion**

This study examined only one mathematics immersion program, but such programs have become an increasingly popular form of professional development. In order to better understand the impact of these programs, examining several different ones would be of value. Ideally, several studies in parallel would assess a number of programs across common areas, such as impact on teachers' conceptions of student learning, conceptions of mathematics as a discipline, affective responses, content knowledge, and, ultimately, classroom practice. Examining several different programs in even one of these areas would be of significant value. In order to accomplish such a task, the belief survey could be refined as described above, and the task-based interviews standardized. Examining several different programs could help answer many of the questions that have been raised regarding the effectiveness of this professional development model (Proulx & Bednarz, 2009). In addition, it could permit comparison between mathematics immersion and other professional development models. In particular, by applying the same research techniques to professional development programs not based on mathematics immersion, the differences between teacher responses could be outlined and the most impactful aspects of professional development programs defined.

Along with comparing results across several different programs, it will be important to assess teacher conceptions and classroom practice over a long period of time. This study was limited to one follow-up visit during the semester immediately following the RLE summer program, but, in order to fully understand

the impact of such a program, research should continue for at least two to three years after the conclusion of the program. Doing so would allow researchers to see whether observed changes persist and become part of a teacher's practice permanently or if teachers eventually revert to their previous practices. Teachers involved in this study (Jennifer and Deborah, for instance) admitted that time constraints prevented them from trying all the new classroom structures and activities they were interested in implementing, but that they hoped, over the course of the next few years, to gradually implement more. Thus, long-term follow-up would be a great benefit to the education community's understanding of the value and impact of mathematics immersion as a professional development model.

No matter the format, more investigation into these programs is necessary for understanding their impact on and value for teachers. This study makes strides toward that goal, and, in particular provides rich descriptions of participants' experiences in the program, which offers insight into the way teachers interpret and respond to their experiences. It does not, however, give long-term or large-scale data that the research literature focusing on these types of programs lacks.

### **Teacher Learning Experiences and Beliefs**

One of the significant conclusions that emerged from this study was that the construction of parallels between teachers' own learning experience in the RLE program and the students' expected learning trajectories was the most significant factor affecting teachers' conceptions of student learning. Every case-

study teacher, over the course of the three interviews, increasingly stressed those aspects of learning that were most significant to their own learning. That is, the conceptions of student learning observed here were most powerfully impacted by teachers drawing parallels between their own learning processes and the learning processes of their students. The degree to which this is true for teacher learning at large, however, remains unclear. There is some evidence to suggest that a teacher's school learning experience plays a significant role in determining their conceptions of student learning (Clark and Person, 1986; Raymond, 1997; Thompson, 1992; Tzur, et al., 2001), but the ways teachers use learning experiences that take place during their careers to make sense of student learning are less well-established. It is possible that the phenomenon observed during this RLE program was due in part to the fact that teacher learning followed the mathematics research trajectory. As established in Chapter II, such a trajectory bears substantial similarities to the student learning process. That their learning proceeded along such a trajectory, even if they were unaware of the similarities, could have played a role in the degree to which they likened their experience to that of their students.

This study provides a starting point for understanding how teachers make use of a learning experience when considering their own students' learning, but leaves a number of questions unanswered. In order to gain a detailed understanding of this topic, it will be necessary to investigate teachers in a variety of learning environments. A standardized protocol for discussing their conceptions of student learning, perhaps based on the task-based interview

protocol used here, could provide a great deal of useful data for comparison. Furthermore, long-term commitment to following teachers back to their classroom would provide data regarding the lasting impact of any observed changes.

Over the long-term, such a program of research could offer insight into the types of teacher education initiatives that impact teachers' conceptions and the ways in which that happens. Moreover, as research programs investigate the most effective ways of teaching students, research into the relationships between teacher learning, teacher conceptions of student learning, and teacher practice could outline provide valuable instruction for the design and implementation of teacher education initiatives.

## **Conclusion**

### **Some Concluding Remarks**

This study was motivated by the observation that mathematics research had striking similarities to student learning, enough so to motivate the question of whether experience with research might change the way a teacher thinks about student learning. By investigating teachers involved in a program that simulates mathematics research, I was able to observe the resiliency of their primary beliefs and the way the experience did shift their peripheral beliefs in ways that mirrored the major aspects of their experience. However, with one exception, the changes did not come about because they saw mathematics research and student learning as similar. Rather, the teachers' personal learning experiences were the primary motivating factor.

This study makes several valuable contributions to the existing body of research on teacher education, mathematics immersion programs, teacher conceptions, and the interaction between these different domains. In particular, the results suggest that teachers shifted peripheral beliefs as their existing conceptions accommodated their experience in the program, leading them to draw parallels between that experience and their students' learning. It also described teachers shifting their conceptions of mathematics research as they participate in a research-like experience. Their descriptions of mathematics research mirrored the major aspects of their learning environment, including exploration of multiple problems prior to conjecturing. Furthermore, they did note similarities between mathematics research and student learning, but, for the most part, these similarities were not the prime motivation behind the shifts in conceptions of student learning. Expecting that student learning would parallel their own was the principal motivator behind those changes.

This research also suggests some questions that should be explored further regarding the role of teacher learning in shaping conceptions and the impact of mathematics immersion programs at large. An understanding of these phenomena can only emerge through the convergence of a multitude of research studies and perspectives. This is one step, one contribution, toward answering the many questions surrounding these topics in mathematics education.

In the end, this study addressed the question posed at the outset, illustrating how a mathematics research experience impacted teachers' conceptions of student learning. As with many topics in education or



investigations involving human subjects, the answers are complex and nuanced. Still, the conclusions detailed above are clear, and contribute some small piece of understanding regarding teacher learning – a foundational issue for the betterment of mathematics education at large.

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## APPENDIX A

### BELIEF INVENTORY

*The category number (I, II, or III), as well as a code indicating whether the item was positively or negatively stated (P for positively state, N for negatively stated) are included in the right-most column. These were not included on the belief survey distributed to the study participants.*

NAME \_\_\_\_\_

**Please respond to each of the following:**

How many summers (counting this one) have you participated in this program?

How many years have you been teaching mathematics?

How many years have you been in your current position?

What mathematics courses have you taught in the last two years?



Please read each of the following items carefully and indicate your agreement or disagreement on a scale of 1 to 5 by circling the corresponding number (1 = Strongly Disagree, 5 = Strongly Agree). In all items, interpret “student” to be a generic term referencing an “average” student at the grade level you teach.

	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree	
1) It is unrealistic to model a mathematics classroom on the behaviors of mathematicians	1	2	3	4	5	N, II
2) Most students can figure out a way to solve many mathematics problems without the help of their teacher	1	2	3	4	5	P, I
3) When mathematicians “do mathematics”, they are doing something fundamentally different that when students “do mathematics”	1	2	3	4	5	N, II
4) Students extend their current knowledge to solve types of problems they’ve never encountered before	1	2	3	4	5	P, III
5) Students shouldn’t be asked to solve problems if they haven’t already seen the procedures for doing so	1	2	3	4	5	N, III

	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree	
6) The process of learning mathematics is always the same, regardless of the level or type of content	1	2	3	4	5	P, II
7) Students need to be given exact procedures for solving problems in mathematics	1	2	3	4	5	N, I, III
8) Students can often figure out relationships between mathematical topics without being shown	1	2	3	4	5	P, I
9) Learning mathematics and researching mathematics involve the same amount of ambiguity	1	2	3	4	5	P, II
10) Student approaches to problems that differ from their teacher's approaches should be encouraged	1	2	3	4	5	P, I, III
11) Students need to be shown several similar examples before solving problems on their own	1	2	3	4	5	N, I

	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree	
12) Mathematics learning depends more on the student than it does on the teacher	1	2	3	4	5	P, I
13) It is better for long, open-ended problems to be broken up into manageable pieces	1	2	3	4	5	N, III
14) Students need a model example to follow when solving problems	1	2	3	4	5	N, III
15) Students learn best when they discover how to solve problems on their own	1	2	3	4	5	P, I
16) One goal of my math courses is to help students think like mathematicians	1	2	3	4	5	P, II
17) The ways students think when learning mathematics and the ways mathematicians think when doing mathematics are different	1	2	3	4	5	N, II

	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree	
18) A teacher shouldn't test or quiz students with problems the students haven't already seen (other than changing the numbers around)	1	2	3	4	5	N, III
19) Experience with mathematics research is helpful for teaching mathematics	1	2	3	4	5	P, II
20) It is appropriate to expect students to figure out a way to solve problems without the help of their teachers	1	2	3	4	5	P, III
21) Mathematicians and mathematics students are both creating mathematics for themselves	1	2	3	4	5	P, II
22) In order to be a good problem solver, it is important for students to follow directions	1	2	3	4	5	N, I
23) Problems assigned to students in mathematics classes should have an obvious solution procedure(s)	1	2	3	4	5	N, II

	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree	
24) Students can explore mathematics in many of the same ways that mathematicians do even if the content is less complex	1	2	3	4	5	P, III
25) Students learn mathematics best by closely attending to teachers' explanations	1	2	3	4	5	N, I
26) Students should be expected to come up with original insights	1	2	3	4	5	P, III
27) The thought processes involved in learning high school mathematics and researching mathematics are different	1	2	3	4	5	N, II
28) In order to be successful in mathematics, a student must be a good listener	1	2	3	4	5	N, I

**If you would like to clarify, expound upon, or discuss any of your responses to the above items, please do so in the space below.**

## APPENDIX B

### INTERVIEW PROTOCOLS

#### INTERVIEW PROTOCOL (Interview 1)

Introduce self again, remind about digital recorder, establish a time to continue the interview if necessary.

##### **Introductory Questions**

- Tell me about your math background.
  - What were your classes like?
  - When did you decide you might want to teach math?
- What classes do you teach?
  - What content do you cover?
  - What do you enjoy/not enjoy about these classes?
  - How would you describe the classroom atmosphere you try to create?
- What kinds of professional development have you participated in?
  - Tell me about last summer. How did you feel about that experience? What did it do for you?

##### **Lesson Planning/Hypothetical Learning Trajectories**

- What curriculum are you using in your algebra class? Do you develop your own lessons or go off of the book?
  - If teacher plans their own lessons, proceed. If not, start with the sample lesson (see attached).
- Describe what I'm looking for:
  - How you would structure this lesson (even if it's over a few days)
  - The reasons you choose to do it that way
  - How you expect student learning to proceed as they move through it.
    - A good question to ask: What do you expect the student to be thinking here?

- Note that I may interject with questions just to probe a bit, and that there is no such thing as too trivial a detail – I want to know all the things you’re thinking as you put it together.
  - Also, I’m not at all looking to judge or assign value to the lessons, I just want to know about your thinking.
- Start with Topic 1: Solving equations (or Solving systems of equations)
  - Present the summary of the lesson (briefly walk through it if necessary, projecting no judgment.
  - Ask if they think it is correctly situated in the curriculum.
  - Ask teacher to walk you through how they would teach it, writing down their lesson plan as they go and “thinking aloud”
  - After the lesson is planned, ask them to walk you through how they expect student learning to proceed (what will be going on in the heads of the students?)
- Repeat with Topic 2: Defining Functions
- What would you need to see in order to consider this lesson to have been effective?
- Thank for time and help and remind them of the next steps.

### **To Keep in Mind**

- “Davidson questioning” - repeating last phrase as a question.
- WAIT for answers – don’t fill blank space – silence in your friend.
- Minimize WHY questions.
- Don’t project judgment – STAY NEUTRAL
- Suspend the interview after an hour and revisit at another time.

## INTERVIEW PROTOCOL (Interview 2)

Remind about digital recorder, establish a time to continue the interview if necessary.

### Introductory Questions

- Tell me about your project. What did you learn while you were doing it?
  - Do you think it will be useful to you? Why or why not?
  - How did you divide the work?
  - What prompted you to look where you did?
- Try to understand if they were surprised by what they were able to do. Self-efficacy.
- Describe the process of mathematics research as you see it.
  - Do you think your experience was representative of math research?
- Did you think about your students while doing this?
  - If so, what about them?

### Lesson Planning/Hypothetical Learning Trajectories

- Remind what I'm looking for:
  - How you would structure this lesson (even if it's over a few days)
  - The reasons you choose to do it that way
  - How you expect student learning to proceed as they move through it.
    - A good question to ask: What do you expect the student to be thinking here?
- Note that I may interject with questions just to probe a bit, and that there is no such thing as too trivial a detail – I want to know all the things you're thinking as you put it together.
  - Also, I'm not at all looking to judge or assign value to the lessons, I just want to know about your thinking.
- Start with Topic 1: Solving equations (or Solving systems of equations)
  - Present the summary of the lesson from previous interview (briefly walk through it if necessary, projecting no judgment).
  - Ask if they think it is correctly situated in the curriculum.
  - Ask teacher to walk you through how they would teach it, writing down their lesson plan as they go and "thinking aloud"
  - After the lesson is planned, ask them to walk you through how they expect student learning to proceed (what will be going on in the heads of the students?)
- Repeat with Topic 2: Defining Functions



- What would you need to see in order to consider this lesson to have been effective?
- Thank for time and help. Establish means of further communication and try to find a tentative time for class observation and interview #3.

## **INTERVIEW PROTOCOL (Interview 3)**

Remind about digital recorder, establish a time to continue the interview if necessary.

### **Introductory Questions**

- Discuss the lesson I've just observed, ask why the teacher made the various choices they made. (These questions cannot be developed in advance). Focus on what conceptions of student learning motivated behavior.
- Have you thought much about your research project since you've returned to the classroom?
- How do you think your teaching this year compares to last year? (Explore this)
- Explore how (if at all) they relate their experience over the summer to their work in the classroom (specific questions to be formulated on an ongoing basis as first two interviews are analyzed).

### **Lesson Planning/Hypothetical Learning Trajectories**

- Remind what I'm looking for:
  - How you would structure this lesson (even if it's over a few days)
  - The reasons you choose to do it that way
  - How you expect student learning to proceed as they move through it.
    - A good question to ask: What do you expect the student to be thinking here?
- Note that I may interject with questions just to probe a bit, and that there is no such thing as too trivial a detail – I want to know all the things you're thinking as you put it together.
  - Also, I'm not at all looking to judge or assign value to the lessons, I just want to know about your thinking.
- Start with Topic 1: Solving equations (or Solving systems of equations)
  - Present the summary of the lesson from previous interview (briefly walk through it if necessary, projecting no judgment).
  - Ask if they think it is correctly situated in the curriculum.
  - Ask teacher to walk you through how they would teach it, writing down their lesson plan as they go and "thinking aloud"

- After the lesson is planned, ask them to walk you through how they expect student learning to proceed (what will be going on in the heads of the students?)
- Repeat with Topic 2: Defining Functions
- What would you need to see in order to consider this lesson to have been effective?
- Thank for time and help. Establish means of further communication.

### **To Keep in Mind**

- “Davidson questioning” - repeating last phrase as a question.
- WAIT for answers – don’t fill blank space – silence is your friend.
- Minimize WHY questions.
- Don’t project judgment – STAY NEUTRAL

### **To Keep in Mind**

- “Davidson questioning” - repeating last phrase as a question.
- WAIT for answers – don’t fill blank space – silence is your friend.
- Minimize WHY questions.
- Don’t project judgment – STAY NEUTRAL

## APPENDIX C

### SAMPLE LESSONS FROM INTERVIEWS

#### **Lesson Outline – Solving Equations**

*\*Adapted from section 2.2 in Benson, et al. (1991). Algebra 2 and Trigonometry, Evanston, IL: McDougal, Littell, & Company.*

#### **Objectives**

Students will be able to solve linear equations using algebraic operations.

\*\*\*This sample lesson identifies the following goals:

*After studying this section, you will be able to:*

- Recognize equivalent equations
- Solve equations by using properties of equality
- Express solutions of equations as ordered pairs

#### **Assumed Prior Content**

- Definition of equality
- Use of variables
- Manipulation of Algebraic Expressions

**Open with this problem (ask students to solve it and explain how they did so):**

Athan needs to rent a car. Dents Rent-a-Car charges \$45 plus 12 cents a mile. Trust Us Rent-a-Car charges \$60 plus 9 cents a mile. Athan quickly computed his cost and found that the cost would be the same at either company. How far was Athan planning to drive?

**Define (on the board):**

Equations with the same solution(s) are called *equivalent equations*. For example,  $x + 6 = 17$ ,  $x + 8 = 19$ , and  $x = 11$  are equivalent equations because all have the same solution, 11.

**To solve an equation, we use this general procedure (Define on the board):**

Write a series of equivalent equations until you arrive at an equivalent equation of the following form: variable = number

**Provide “algebraic properties . . . rules that tell us what we can and cannot do” (on board):**

- **Addition and Subtraction Properties of Equality**  
If  $a = b$ , then  $a + c = b + c$  and  $a - c = b - c$   
We can add the same quantity to, or subtract the same quantity from, each side of an equation and the resulting equation is equivalent to the original equation.
- **Multiplication and Division Properties of Equality**  
If  $a = b$ , then  $ac = bc$  and (if  $c \neq 0$ )  $\frac{a}{c} = \frac{b}{c}$   
We can multiply or divide each side of an equation by the same numerical quantity (provided that the quantity is not zero) and the resulting equation is equivalent to the original equation.
- **Zero Product Property**  
If  $a \cdot b = 0$ , then  $a = 0$  or  $b = 0$   
If the product of two or more numbers is zero, at least one of the numbers must be zero.

**State to the whole class (on board):**

To solve an equation means to find all values of the variable that make the equation a true statement.

**Do an example on the board (illustrate and explain each step):**

$$\begin{array}{r}
 3x - 7 = 2x + 4 \\
 -2x \quad -2x \\
 \hline
 x - 7 = 4 \\
 +7 \quad +7 \\
 \hline
 x = 11
 \end{array}$$

**Check work by substituting 11 for  $x$  on each side**

**Show to the whole class (on board):**

Often listed in a table, solutions of an equation in two variables, such as  $x$  and  $y$ , are also shown as *ordered pairs*, with  $x$  as the first number of each pair and  $y$  as the second number.

**Show example:** using  $y = 2(10 - x)$

**Sample Problems (Do as a class):**

Two runners live 33 miles apart. Jim runs 8 miles per hour, and Swett runs 6 miles per hour. If Swett starts running toward Jim’s house and Jim

starts running toward Swett's house 2 hours later, how long will Jim have been running when they meet?

The area of a rectangle  $A$  is equal to the area of rectangle  $B$ . Find the dimensions of each rectangle.

**Problem Set** (have individuals work on examples out of the book)

## Lesson Outline – Using Algebra to Solve Systems of Equations

\*Taken from section 5.7 in *Math Connections (1b)*

### Identified Learning Outcomes

After studying this section, you will be able to:

- Use algebra to solve a system of two equations both of the form  $y=mx+b$
- Identify the independent and dependent variable
- Determine if a pair of values is a solution to a system of two equations in two variables.

### Assumed prior knowledge

- Understanding of linear equations (graphing, linear relationships)
- Solving a linear equation
- Solving problems where a line intersects a horizontal line
- Laws of Algebra

Previous section shows how a linear equation can be thought of as the intersection of a horizontal line and some other line.

Begin with Electric Company rate schedule example from previous section ( $T=0.09u+8.5$  is the old rate schedule and  $T=0.10u+6$  is the new rate schedule). Use this to extend previous section in order to consider where these two non-horizontal lines intersect.

Show how we can set the two  $T$ 's equal to each other to get the equation  $0.09u+8.5=0.10u+6$ .

Demonstrate subtracting one  $u$  term from both sides and then proceeding with the previous algorithm.

Show the graph of the system, and the intersection point.

Define *dependent variable* and *independent variable*

Discuss what a *solution* of a system of linear equations is. Use an example system to try some points and see if they are solutions.

Practice solving systems of linear equations using algebra.

## Lesson Outline – Defining Functions

*\*\*Adapted from section 6.1 in Berlinghoff, Sloyer, & Hayden (2000). Math Connections, Book 1b, Armonk, NY: It's About Time, Inc.*

### Identified Learning Outcomes

After studying this section, you will be able to:

- Identify and explain functions in real world situations
- Describe real world relationships using the language of functions
- Find images for particular domain elements when given a function described in words, by a pattern, or with a table.

### Assumed prior knowledge

- Linear equations
- Representation using variables, independent and dependent variables for graphing and lines.

### **Begin with FBI's fingerprint database as an example (put on board):**

Each fingerprint leads to only one person, but since people have more than one finger, more than one fingerprint may lead to the same person. Identify this as an example of a *function*

### **Define (on the board)**

A *function* is a process that relates each thing in a first set to exactly one thing in a second set. The first set is called the *domain* and the second set is called *range* (Identify these in the above example)

### **Discuss as a class:**

What are the everyday English meanings of *function* - highlight how it often indicates dependence.

### **Offer a second example (on board):**

ZIP codes assign each piece of mail to a specific post office location.

### **Ask the entire class:**

What are other possible examples (e.g. The volume of a quantity of gas is a function of the pressure put on it; The size of a colony of bacteria is a function of the time it has been growing; The pay of a cook at Burger King in a function of the number of hours he or she works) – do the white pages in the phone book form a function? Why or why not?

### **Remind students of earlier definitions (on the board):**

With the "real world" examples, express them as functions, and identify domain and range.



**Show function notation (on board):**

We can abbreviate long descriptions using the language of functions. We name the function using a letter (such as  $f$ ) and write  $f(\quad)$ . Use this notation for previous examples (e.g.,  $f(\text{fingerprint}) = \text{John Doe}$ )

**Identify some functions using arrow diagrams, practice finding images using notation (have class work on them in groups)**

**Use arrow diagrams (on board)**

Present other situations and determine if they are functions.

Ex:

**Show (on board)**

We might use algebraic notation to write a rule for a particular function (ex.  $A(x)=s^2$  is the area function for squares, where  $s$  is the length of a side). Look at some examples:

$A(r) = \pi r^2$  – Assigns the area of a circle given the radius

**Define (on the board):**

When a function is defined by a formula, the symbol that stands for the domain element is the *independent variable*, and the symbol that stands for its image is the *dependent variable*.

**Ask the class:**

What are the independent and dependent variables in the previous examples?

**Have class work on problems individually.**

## APPENDIX D

### INSTITUTIONAL REVIEW BOARD APPROVAL LETTER

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# University of New Hampshire

Research Integrity Services, Office of Sponsored Research  
Service Building, 51 College Road, Durham, NH 03824-3585  
Fax: 603-862-3564

25-Jun-2009

Abel, Todd  
Math and Statistics, Kingsbury Hall  
9 Lincoln Ave #9B  
Newmarket, NH 03857

**IRB #:** 4625

**Study:** How a Mathematics Research Experience Impacts Teachers' Conceptions of Student Learning

**Approval Date:** 25-Jun-2009

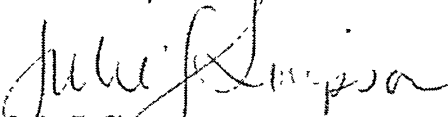
The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Exempt as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 101(b). Approval is granted to conduct your study as described in your protocol.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, *Responsibilities of Directors of Research Studies Involving Human Subjects*. (This document is also available at <http://www.unh.edu/osr/compliance/irb.html>.) Please read this document carefully before commencing your work involving human subjects.

Upon completion of your study, please complete the enclosed Exempt Study Final Report form and return it to this office along with a report of your findings.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or [Julie.simpson@unh.edu](mailto:Julie.simpson@unh.edu). Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,



Julie F. Simpson  
Manager

cc: File  
Graham, Karen